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1. We start with an example. It is well known [1], p. 185 that every universal algebra can be embedded into certain semigroup. For instance, according to this theorem, in case of the groupoid determined by the table

	o	a	b
a	a	a	b
b	a	a	a

(1)

we have the following assertion:

(2) There exists a set  $S \supset \{a, b\}$ , an operation  $\star : S^2 \rightarrow S$  and  $c \in S$  such that  $(S, \star)$  is a semigroup and the equality

$$xoy = (c \star x) \star y$$

is true for all  $x, y \in \{a, b\}$ .

We describe a construction of such a semigroup  $(S, \star)$ , whose elements will be generated by  $a, b$  and one new element  $c$ . In fact this construction is an illustration of some general ideas we are going to describe after that.

At first extend the set  $\{a, b\}$  by a new element  $c$ .

c say. Further, by  $\text{Term}(a,b,c,*)$ , where  $*$  is a binary operation symbol, we denote the set of all terms built up from  $a,b,c$  and  $*$  (without any variables).

In connection with (2) form the following set  $H(=(3) \cup (4) \cup (5))$ :

(3)  $(c*a)*a = a, (c*a)*b = b, (c*b)*a = a, (c*b)*b = a$ ,

i.e. all equalities of the form

$$xoy = (c*x)*y \quad (x,y \in \{a,b\})$$

(4)  $a \neq b$

(5)  $(t_1*t_2)*t_3 = t_1*(t_2*t_3)$  for all  $t_i \in \text{Term}(a,b,c,*)$ .

Further let  $Ax(=)$  be the set of the following formulas (the equality axioms):

(6)  $t_1=t_1, t_1=t_2 \Rightarrow t_2=t_1, t_1=t_2 \wedge t_2=t_3 \Rightarrow t_1=t_3$   
 $t_1 = t_2 \wedge t_3 = t_4 \Rightarrow t_1*t_3 = t_2*t_4 \quad (t_i \in \text{Term}(a,b,c,*)$ .

Obviously any normal model<sup>1)</sup> of the set  $H \cup Ax(=)$  (of the set  $H$  as well) determines the required semigroup  $(S, \star)$ .

We point out that the set  $H \cup Ax(=)$  is a set of basic<sup>2)</sup> Horn formulas.

Besides this, in algebra, there are many other problems which can be expressed by means of certain sets of basic Horn formulas. For instance, the problems of isomorphic embeddings, constructions of free algebras and so on.

We describe one method which is often useful for solving such problems. This method is partly original. The paper is deeply connected with [2].

At first we construct the required semigroup  $(S, \star)$ .

In the first step we search for some quasi-algebra [2], which follows from the set  $H \cup Ax(=)$ . To achieve this, define the equivalence relation  $\sim$  (of  $\text{Term}(a,b,*)$ ) as follows:

(7)  $t_1 \sim t_2$  iff  $H \cup Ax(=) \vdash t_1 = t_2$ .

Using this definition it is easy to prove that:

Each member  $t$  of  $\text{Term}(a,b,c,*)$  is equivalent to at least one term  $t'$  having the property:

If  $t'$  contains<sup>3)</sup> the symbol  $c$ , then after every occurrence of  $c$  there is at most one occurrence of  $a$  or  $b$ .

Let  $M$  be the set of all  $t'$ , where  $t \in \text{Term}(a,b,c,*)$ . The members  $t'$  of  $M$  are called markers. For instance

$$a, aa, c, aca, caac \in M, \text{ but } cab \notin M.$$

Let  $m_1, m_2$  be any two markers and let  $\overline{m_1 m_2}$  be the marker which is obtained from the term  $m_1 m_2$  by using the substitutions (see (3)):

$$caa \rightarrow a, cab \rightarrow b, cba \rightarrow a, cbb \rightarrow a.$$

It is easy to prove that  $\overline{m_1 m_2}$  is unique determined by  $m_1, m_2$ . For example, if  $m_1 = caccb, m_2 = abca$ , then

$$\begin{aligned} \overline{m_1 m_2} &= \overline{caccbabca} = \overline{cacabca} \\ &= \overline{cabca} = bca \\ &\quad cba \rightarrow a \\ &\quad = cabca = bca \\ &\quad cab \rightarrow b. \end{aligned}$$

Denote by  $Q$  the set of all equalities of the form:

$$m_1 m_2 = \overline{m_1 m_2}.$$

This set is a quasi-algebra.

In the second step we check whether the quasi-algebra  $Q$  is incontractible. Using the way described in [2] it is easy to see that  $Q$  is incontractible, i.e. that

$$\text{not } Q \cup Ax(=) \vdash m_1 = m_2$$

where  $m_1, m_2$  are some two different members of the set  $M$ . The set  $Q$  is a consequence of the set  $H \cup Ax(=)$ . That is why the equivalence

$$(8) \quad H \cup Ax(=) \text{ equiv. } H \cup Ax(=) \cup Q$$

is true.

In the third step we replace each term  $t$  of each formula of the set  $H$  by the corresponding marker  $t'$ . In such a way from the set  $H$  we obtain a new set  $H_Q$  so called the reduct of the set  $H$  modulo the quasi-algebra  $Q$ .

Obviously the equivalence

$$(9) \quad H \cup Ax(=) \cup Q \text{ equiv. } H_Q \cup Ax(=) \cup Q$$

is true. From (8), (9) it follows that

$$(10) \quad H \cup Ax(=) \text{ equiv. } H_Q \cup Ax(=) \cup Q.$$

We now "calculate"  $H_Q$ . For the set (3) we have  $(3)_Q = \{a=a, b=b, a=a, a=a\}$ .

It is not quite easy to find  $(5)_Q$ , i.e. the set

$$(5)_Q = \{\overline{m_1 m_2} m_3 = \overline{m_1 m_2} m_3 \mid m_1, m_2, m_3 \in M\}.$$

The result is

$$(5)_Q = \{m = m \mid m \in M\}.$$

In other words the quasi-algebra  $Q$  satisfies the associative law. For the set (4) we have

$$(4)_Q = \{a \neq b\},$$

because  $a, b$  are markers and therefore:  $a'$  is  $a$ ,  $b'$  is  $b$ .

As  $H_Q = (3)_Q \cup (4)_Q \cup (5)_Q$  we have the following result (for  $H_Q$ ):

$$H_Q = \{a \neq b\} \cup \{m = m \mid m \in M\}.$$

For the set  $H_Q$  we note the following:

(11) The reduct  $H_Q$  contains neither any equality  $m_1 = m_2$  between two different<sup>4)</sup> markers nor any member of the form  $m \neq m$ .

From this and (10) it follows the following equivalence

$$(12) \quad H \cup Ax(=) \text{ equiv. } Ax(=) \cup Q \cup \{a \neq b\}.$$

The set  $Q \cup \{a \neq b\}$  is called extended quasi-algebra. Generally, if  $Q$  is any quasi-algebra and  $D$  is any set of some formulas of the form

$$m_1 \neq m_2 \quad (m_1, m_2 \text{ are different markers}),$$

then the set  $Q \cup D$  is called extended quasi-algebra.

From the equivalence (12) we conclude that the required semigroup is determined by any model of the extended quasi-algebra  $Q \cup \{a \neq b\}$ .

Obviously this extended quasi-algebra is consistent (since  $Q$  is incontractible and  $a, b$  are two different markers as well). Therefore the set  $Q \cup \{a \neq b\}$  has at least one model. One of models  $(S, \star)$  (in fact the marker model) is determined as follows:

1<sup>o</sup> The set  $S$  is equal to the set  $M$  of all markers.

$2^0$  The operation  $\star$  is defined by

$$m_1 \star m_2 = \overline{m_1 m_2}$$

In such a way we have completed the construction of the required semigroup  $(S, \star)$ .

Of course to prove the assertion (2) we do not need effectively to construct one semigroup  $(S, \star)$  satisfying (2). Namely, to prove (2) it is sufficient to prove the existence of any model of the set  $H \cup Ax(=)$ , i.e. the consistency of that set.

It is easy to see that the consistency condition is equivalent to the following condition

$$(13) \quad \text{Not } H \cup Ax(=) \vdash a = b$$

i.e. that the terms  $a, b$  (all members of the given groupoid (1)) are not equivalent.

Suppose the opposite, i.e.

$$H \cup Ax(=) \vdash a = b$$

which is equivalent to 5)

$$H \vdash \overline{Ax(=)} \quad a = b,$$

i.e. the equality  $a=b$  can be derived from the set  $H$  using the equational logic. Let

$$(14) \quad a = t_1, \quad t_1 = t_2, \dots, t_k = b$$

be one of such derivations. By the substitutions

$$(c \star a) \star a \rightarrow (a o a), \quad (c \star a) \star b \rightarrow (a o b),$$

$$(c \star b) \star a \rightarrow (b o a), \quad (c \star b) \star b \rightarrow (b o b)$$

from the derivation (14) we obtain a derivation of the equality  $a=b$  in the given groupoid, which is impossible. Consequently the condition (13) is proved.

2. We are now going to study the general case. Let  $\Omega$  be a given set of operation symbols and  $\Gamma$  a given set of constants. By Term  $(\Omega, \Gamma)$  denote the set of all terms built up from  $\Omega$  and  $\Gamma$  (but without variables). Further let  $H$  be a set of some (basic Horn) formulas, i.e. formulas of the form

$$\phi_1, \neg \phi_1, \phi_1 \wedge \dots \wedge \phi_k \Rightarrow \phi_{k+1}, \quad \phi_1 \wedge \dots \wedge \phi_k \Rightarrow \neg \phi_{k+1}$$

( $k = 1, 2, \dots$ ), where  $\phi_i$  are of the form

$$t_1 = t_2 \quad (t_1, t_2 \text{ Term } (\Omega, \Gamma)).$$

We also suppose that certain problem  $\mathcal{P}$ , like the considered, is expressed by the set  $H$ . We distinguish two cases:

To solve the problem  $\mathcal{P}$  means

$1^0$  to prove that  $H$  has a model,

or  $2^0$  to construct a model of  $H$ .

The considered problem  $\mathcal{P}$  belongs to the case  $1^0$ .

As a matter of fact this problem is expressed by the set  $H (= (3) \cup (4) \cup (5))$  but with quantifiers  $(\exists S) (\exists *: S^2 \rightarrow S)$  as prefixes, i.e.  $\mathcal{P}$  is exactly expressed by the "formula"

$$(\exists S) (\exists *: S^2 \rightarrow S) H.$$

Generally in case  $1^0$  the problem  $\mathcal{P}$  is expressed by  $H$ , preceding by some existential quantifiers.

In this case to solve the problem  $\mathcal{P}$  it is sufficient to prove the consistence of the set  $H$ . In the literature there are many particular ideas about it. For instance, we can often use the ideas which are similar to those used in the considered problem.

In this paper we are more interested in case 2°.

We sketch, step by step, one solving method similar to the method used in the considered example.

In the first step we search for some quasi-algebra which follows from the set  $H \cup Ax(=)$ . To achieve this we use the definition of the type (7).

In the second step we check whether the quasi-algebra  $Q$  is incontractible using the way described in [2].

If  $Q$  is contractible, then there are some equalities of the form

$$m_1 = m_2 \quad (m_1, m_2 \text{ are different markers})$$

which follows from  $Q \cup Ax(=)$ . Then using such equalities we reduce the set  $M$  of all markers to some its proper subset  $M_1$ , and form a corresponding quasi-algebra,  $Q_1$  say.

We proceede in such a way, until we obtain some quasi-algebra which is incontractible.

In the third step we form the set  $H_Q$ , the reduct of  $H$  modulo  $Q$ , by replacing each term  $t$  of every formula of the set  $H$  by the corresponding equivalent marker  $t'$ .

We also replace the formulas of the form

$$m = m \wedge \dots \wedge n = n \Rightarrow m_1 = m_2, \quad m = m \wedge \dots \wedge n = n \Rightarrow m_1 \neq m_2$$

$$m_1 = m_2 \Rightarrow m \neq n \quad (m_1, m_2, m, n \text{ are markers})$$

by

$$m_1 = m_2, \quad m_1 \neq m_2, \quad m_1 \neq m_2$$

respectively

Then, similarly as in the example (see (10)) the equivalence

$$(15) \quad H \cup Ax(=) \text{ equiv. } H_Q \cup Ax(=) \cup Q$$

is true. The members of the set  $H_Q$  are of the form

$$\phi_1, \neg\phi_1, \phi_1 \wedge \dots \wedge \phi_k \Rightarrow \phi_{k+1}, \phi_1 \wedge \dots \wedge \phi_k \Rightarrow \neg\phi_{k+1}$$

( $k = 1, \dots$ ) where  $\phi_i$  are some equalities of the form  $m_1 = m_2$  ( $m_1, m_2$  are markers).

For the set  $H_Q$  there are two possibilities

The sentence (11) is either true or false.

If (11) is false and  $H_Q$  has some member of the form  $m \neq m$ , then the set  $H$  is inconsistent and consequently there is no universal algebra which is a solution of the problem  $\mathcal{P}$ .

If (11) is false and certain equality  $m_1 = m_2$  between two different markers is a member of the reduct  $H_Q$ , then using all such equalities<sup>6)</sup> we reduce the quasi-algebra  $Q$  and go back to the second step.

If the sentence (11) is true we form the corresponding extended quasi-algebra  $Q \cup D$  ( $D$  is the set of all members of  $H_Q$  which are of the form  $m_1 \neq m_2$ , where  $m_1, m_2$  are different markers). Then from (15) we conclude the following equivalence

$$(16) \quad H \cup Ax(=) \text{ equiv. } Q \cup D \cup Ax(=) \cup J,$$

where  $J$  is a set of some implications of the form

$$(17) \quad m_1 = n_1 \wedge \dots \wedge m_k = n_k \Rightarrow m_{k+1} = n_{k+1}$$

$$m_1 = n_1 \wedge \dots \wedge m_k = n_k \Rightarrow m_{k+1} \neq n_{k+1} \quad (k=1, 2, \dots)$$

and  $m_i, n_i$  ( $1 \leq i \leq k$ ) are pairwise different markers.

The marker algebra - whose members are markers and operations are defined using directly the equalities belonging to the quasi-algebra  $Q$  - is a model for  $Q$ , for  $Q \cup D$ , for  $Q \cup D \cup J$  and also a model for  $H$ .

In this way the mentioned method is completely described. We also add the following remarks:

- (i) By the described method the consistency problem for the set H is also solved.
- (ii) The obtained model, i.e. the marker algebra is, in fact, a model of H generated<sup>7)</sup> by the constants belonging to  $\Gamma \cup \Omega$ .

3. At the end we give one problem in which a non basic Horn set of formulas appears but which can be solved in a similar way (searching for an incontractible quasi-algebra Q which is a consequence of the corresponding set H, after that forming the reduct  $H_Q$ , and so on).

Let  $J_2$  be a field with two elements 0, e defined by the tables

(18)	+	0	e	·	0	e
	0	0	e	0	0	0
	e	e	0	e	0	e

and  $x^2 + x + e = 0$  be an equation in x.

The problem is to construct the root field for this equation. We sketch one way of solving.

Let H be the set of the following formulas  
 ((19)  $\cup \dots \cup$  (23))

- (19)  $0+0=0, \quad 0+e=e, \quad e+0=e, \quad e+e=0$
- $0 \cdot 0=0, \quad 0 \cdot e=0, \quad e \cdot 0=0, \quad e \cdot e=e$
- $-0=0, \quad -e=e, \quad e^{-1}=e$

(20)  $0 \neq e$

- (21)  $(x+y)+z=x+(y+z), \quad x+0=x, \quad x+(-x)=0, \quad x+y=y+x$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x \cdot e = x, \quad x \neq 0 \Rightarrow x \cdot x^{-1} = e, \quad x \cdot y = y \cdot x$
- $x \cdot (y+z) = x \cdot y + x \cdot z$

(22)  $a^2 + a + e = 0$

(23)  $0^{-1} = 0$

where  $\underline{a}$  is a new constant symbol and  $x, y, z$  are elements of the set

(24)  $\text{Term}(0, e, a, +, \cdot, ^{-1})$

The formulas (19)  $\cup$  (20) are members of the diagram of the given field  $J_2$ , the formulas (21) are the field axioms, the formula (22) expresses that  $\underline{a}$  is a solution of the given equation and the formula (23) is taken to simplify our consideration<sup>8)</sup>.

Obviously any (normal) model of these formulas determines the required<sup>9)</sup> field.

In this case about the relation  $\sim$ , introduced by (as (7))

$$t_1 \sim t_2 \quad \text{iff} \quad H \cup Ax(=) \vdash t_1 = t_2,$$

we have the following assertion

Each term (a member of the set (22)) is equivalent to one of the following terms (i.e. markers)

$$0, e, a, a+e.$$

For instance

$$0^{-1} \sim 0, \quad \text{since} \quad 0^{-1} = 0 \in H$$

$$a^{-1} \sim a+e, \quad \text{since} \quad a(a+e) \sim e \quad \text{and} \quad H \cup Ax(=) \vdash a \neq 0,$$

which can be proved in the following way

(j)  $H \cup Ax(=) \cup \{a=0\} \vdash e=0$

(jj)  $H \cup Ax(=) \vdash a=0 \Rightarrow e=0$ , from (j)

by the Deduction theorem

$$(jjj) H \cup Ax(=) \vdash e \neq 0 \implies a \neq 0$$

$$(jw) H \cup Ax(=) \vdash a \neq 0, \text{ since } e \neq 0 \in H.$$

One quasi-algebra  $\Omega$ , which is a consequence for the set  $H \cup Ax(=)$ , is determined by the following equalities:

$$(25) \begin{array}{llll} 0+0=0 & 0+e=e & 0+a=a & 0+(ate)=ate \\ e+0=e & e+e=0 & e+a=ate & e+(ate)=a \\ a+0=a & ate=ate & a+a=0 & a+(ate)=e \\ (ate)+0=ate & (ate)+e=a & (ate)+a=e & (ate)+(ate)=0 \end{array}$$

$$(25) \begin{array}{llll} 0 \cdot 0=0 & 0 \cdot e=0 & 0 \cdot a=0 & 0 \cdot (ate)=0 \\ e \cdot 0=0 & e \cdot e=e & e \cdot a=a & e \cdot (ate)=ate \\ a \cdot 0=0 & a \cdot e=a & a \cdot a=ate & a \cdot (ate)=e \\ (ate) \cdot 0=0 & (ate) \cdot e=ate & (ate) \cdot a=e & (ate) \cdot (ate)=a \end{array}$$

$$-0=0, \quad -e=e, \quad -a=a, \quad -(ate) = ate;$$

$$0^{-1}=0, \quad e^{-1}=e, \quad a^{-1}=ate, \quad (ate)^{-1}=ate$$

It is not difficult to prove that  $H_{\Omega}$  is equivalent to the set  $\{e \neq 0\}$ . Further, from the set  $\Omega \cup \{a \neq 0\} \cup Ax(=)$  we obtain the following inequalities

$$(26) \quad e \neq 0, \quad a \neq 0, \quad ate \neq 0, \quad a \neq e, \quad ate \neq e, \quad ate \neq a.$$

Finally we conclude that the set (25)  $\cup$  (26) determines the required field.

We point out that a similar way can be used to construct the root field of any given equation (on some field  $F$ ).

1) i.e. model in which the sign = is interpreted as equality.

- 2) Each member of that set is of the following type  $\phi_1, \neg\phi_1, \phi_1 \wedge \dots \wedge \phi_k \implies \phi_{k+1}, \phi_1 \wedge \dots \wedge \phi_k \implies \neg\phi_{k+1}$  ( $k=1, 2, \dots$ ) where  $\phi_i$  are formulas of the form  $t_1 = t_2$  ( $t_1, t_2 \in \text{Term}(a, b, c, *)$ ).
- 3) We write  $x_1 x_2, x_1 x_2 x_3, \dots$  instead of  $x_1 * x_2, (x_1 * x_2) * x_3, \dots$  respectively.
- 4) different as terms.
- 5) i.e. the equality  $a = b$  can be derived from the set  $H$  by using the equational logic.
- 6) To speed up the algorithm we can also use every equality of the form  $m_1 = m_2$  ( $m_1, m_2$  are different markers), which is a consequence of  $H_{\Omega} \cup Ax(=)$  (by propositional logic).
- Similarly, if a formula of the form  $m \neq m$  is deduced from  $H_{\Omega} \cup Ax(=)$ , then the problem  $\mathcal{P}$  has no solution.
- 7) Other such models are the homomorphic images of the marker algebra corresponding to the congruences of the marker algebra which preserve (i.e. satisfy) the conditions  $D \cup J$ .
- 8) In any field we may use the definition  $0^{-1} = 0$ .
- 9) If  $\underline{a}$  is a solution of the equation  $x^2 + x + e = 0$ , so is  $e + a$ .

#### R E F E R E N C E S

- [1] Cohn, P.M., Universal Algebra, 1965  
 [2] Prešić, S.B., On quasi-algebras and the word problem, Public.de l'Inst.math., T. 26(40), 1979, 255-268  
 Beograd