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GENERALIZING LOGIC PROGRAMMING TO ARBITRARY SETS OF CLAUSES

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Abstract. In this paper, which is a brief version of [3], we state how one can extend Logic Programming to any set of clauses.

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The basic part of Logic Programming, particularly Prolog, in fact deals with the following two inference rules:

(1) $\mathcal{F}, p \vdash p$

(2) $\mathcal{F}, p \vee \neg q_1 \vee \ldots \vee \neg q_k \vdash p \longleftarrow \mathcal{F} \vdash q_1, \ldots, q_k$

(where $\mathcal{F}$ is a set of (positive) Horn formulas and p is any atom, i.e. a propositional letter)

Indeed, the informal meaning of rule (1) is:

An atom p is a consequence of a set of clauses if p is an element of that set.

Similarly for rule (2) we have this meaning:

An atom p is a consequence of a set $\mathcal{F}, p \vee \neg q_1 \vee \ldots \vee \neg q_k$ *(i.e of the set* $\mathcal{F}, q_1 \wedge \ldots \wedge q_k \Rightarrow p$*), if*$q_1, \ldots, q_k$ *are consequences of the set* $\mathcal{F}$.

In the sequel we use the following facts from mathematical logic (see [2]):

(3) *The notion of formal proof in the case of propositional logic (assuming we have chosen some tautologies as axioms, and that* **modus ponens** *is the only inference rule).*

(4) **The Deduction theorem**[1] : $\mathcal{F}, A \vdash B \longleftrightarrow \mathcal{F} \vdash A \Rightarrow B$ *where* $\mathcal{F}$ *is a set of propositional formulas and* A, B *are some such formulas.*

(5) **Completeness Theorem:** *Any propositional formula is a logical theorem if and only if it is a tautology.*

We also use the symbols $\bot, \top$ which can be introduced by the following definitions

$\bot$ stands for $a \wedge \neg a$; $\top$ stands for $a \vee \neg a$

where a is an atom (chosen arbitrarily). Further, let $\mathcal{F}$ be any set of propositional formulas and ψ a formula or one of the symbols $\bot, \top$. Then a **sequent** is any expession of the form $\mathcal{F} \vdash \psi$, with the meaning:

ψ is a logical consequence of $\mathcal{F}$

Lemma 1. *Let $\mathcal{F}$ be any set of propositional formulas not containing the atom p, and let $\phi_1(p), \phi_2(p), ...$ be propositional formulas containing p. Then we have the following equivalences*

(6) (*i*) $\mathcal{F}, \phi_1(p), \phi_2(p), ... \vdash p \longleftrightarrow \mathcal{F}, \phi_1(\bot), \phi_2(\bot), ... \vdash \bot$

(*ii*) $\mathcal{F}, \phi_1(p), \phi_2(p), ... \vdash \neg p \longleftrightarrow \mathcal{F}, \phi_1(\top), \phi_2(\top), ... \vdash \bot$

Proof. First we give proof of the $\longrightarrow$ part of (i). Then, we have the following 'implication-chain':

$\mathcal{F}, \phi_1(p), \phi_2(p), ... \vdash p$

$\longrightarrow$ For some formulas $f_1, ..., f_r$ of $\mathcal{F}$ and some formulas $\phi_{i1}(p), ..., \phi_{is}(p)$ we have: $f_1, ..., f_r, \phi_{i1}(p), ..., \phi_{is}(p), ... \vdash p$
(Finiteness of the propositional proof)

$\longrightarrow$ $\vdash f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p$
(By (4))

$\longrightarrow$ Formula

$$f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p$$

is a tautology
(By (5))

$\longrightarrow$ Formula $f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot$
is a tautology

[1] In fact, only the $\longrightarrow$-part is the deduction theorem. But, the $\longleftarrow$-part is almost trivial.

$\longrightarrow$ Formula

$$f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot$$

is a logical theorem

(By (5))

$\longrightarrow$ Formula

$$f_1, ..., f_r, \phi_{i1}(\bot), ..., \phi_{is}(\bot) \vdash \bot$$

holds.

(By (4))

$\longrightarrow$ $\mathcal{F}, \phi_1(\bot), \phi_2(\bot), ... \vdash \bot$

which completes the proof. Proof of the $\longleftarrow$ part of (i) reads:
$\mathcal{F}, \phi_1(\bot), \phi_2(\bot), ... \vdash \bot$

$\longrightarrow$ For some formulas $f_1, ..., f_r$ of $\mathcal{F}$ and some formulas $\phi_{i1}(\bot), ..., \phi_{is}(\bot)$ we have: $f_1, ..., f_r, \phi_{i1}(\bot), ..., \phi_{is}(\bot), ... \vdash \bot$

(Finiteness of every formal proof)

$\longrightarrow$ $\vdash f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot$

(By (4))

$\longrightarrow$ Formula

$$f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot$$

is a tautology

(By (5))

$\longrightarrow$ Formula

$$f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p$$

is a tautology

$\longrightarrow$ Formula

$$f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p$$

is a logical theorem

(By (5))

$\longrightarrow$ Formula

$$f_1, ..., f_r, \phi_{i1}(p), ..., \phi_{is}(p) \vdash p$$

holds.

(By (4))

$\longrightarrow$ $\mathcal{F}, \phi_1(p), \phi_2(p), ... \vdash p$

which completes the proof of (i).
We have omitted a proof of (ii) because (ii) can be proved in a similar way as (i).
Notice that Lemma 1 can be expressed by the following words:

> *A literal*[2] ψ *is a logical consequence of the given set if and only if the corresponding*[3] *set is inconsistent.*

Now we prove the following lemma.
Lemma 2. *The equivalence*

(7) $$\mathcal{F}, p_1 \vee ... \vee p_k \vdash \bot \longleftrightarrow \mathcal{F} \vdash \neg p_1, ..., \mathcal{F} \vdash \neg p_k$$

(where p_i is any literal)

is true.

Proof. We have the following 'equivalence-chain':
$\mathcal{F}, p_1 \vee ... \vee p_k \vdash \bot$

$\longleftrightarrow$ $\mathcal{F} \vdash (p_1 \vee ... \vee p_k \Longrightarrow \bot)$

(By (4))

$\longleftrightarrow$ $\mathcal{F} \vdash (\neg p_1 \wedge ... \wedge \neg p_k)$

(Using a well-known tautology)

$\longleftrightarrow$ $\mathcal{F} \vdash \neg p_1, ..., \mathcal{F} \vdash \neg p_k$

which completes the proof.
Besides (6) and (7) we emphasize the following obvious equivalences

(8) $$\vdash \top \longleftrightarrow \mathcal{F}, \bot \vdash \bot$$

(9) $$\mathcal{F}, \top \vdash A \longleftrightarrow \mathcal{F} \vdash A$$

(A is a literal or the symbol $\bot$)

[2] A literal is an atom or the negation of an atom
[3] i.e. one of the sets $\mathcal{F}, \phi_1(\bot), \phi_2(\bot), ...$ or $\mathcal{F}, \phi_1(\top), \phi_2(\top), ...$

Suppose now that $\mathcal{F}$ is a given set of clauses and ψ is a literal or $\bot$. Is it possible that using the equivalences (6), (7), (8), (9) one can establish whether or not ψ is a logical consequence of $\mathcal{F}$? In order to answer this we introduce the following inference rules[4]

(R1) $\mathcal{F}, \bot \vdash \bot \longleftarrow \vdash \top$

(R2) $\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \longleftarrow \mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot$

$\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash \neg p \longleftarrow \mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \vdash \bot$

($\phi_i(p)$ is any clause containing p)

(R3) $\mathcal{F}, p_1 \vee \ldots \vee p_k \vdash \bot \longleftarrow \mathcal{F} \vdash \neg p_1, \ldots, \mathcal{F} \vdash \neg p_k$

(where p_i is any literal)

(R4) $\mathcal{F}, \top \vdash A \longleftarrow \mathcal{F} \vdash A$

(A is a literal or the symbol $\bot$)

We emphasize that in the sequel for the set $\mathcal{F}$ we suppose that it does not contain a clause of the form $\ldots q \vee \neg q \ldots$,where q is any atom. Namely, such a formula is equivalent to $\top$, consequently it should be omitted[5]. Similarly, if it happens that by applying rule (R2) some clause becomes equivalent to $\top$ then we will also omit it.

Roughly speaking rules (R1),(R2),(R3),(R4) are used as follows:

> *We start with a question (a sequent) of the form $\mathcal{F} \vdash \psi$ and apply rules (R2),(R3),(R4) several times. If at some step we can apply rule (R1), the procedure stops with the conclusion that ψ is a logical consequence of $\mathcal{F}$. However, if at some step we obtain the sequent $\vdash \bot$ (then $\mathcal{F}$ is an empty set) the procedure stops with the conclusion that ψ is not a logical consequence of $\mathcal{F}$.*

Example 1. Answer the following questions:

1) $p \vdash p$? 2) $p, q \vdash p$? 3) $\vdash p$? 4) $q \vdash p$?

5) $\neg q \vee p, q \vee p \vdash p$? 6) $p, \neg p \vee q \vee \neg r, p \vee \neg q \vee s, p \vee s \vee \neg t \vdash \bot$?

where p, q, r, s, t are atoms.

Answer.

1) Applying (R2) we obtain the sequent $\bot \vdash \bot$ and by (R1) we get the sequent $\vdash \top$ so the answer is: *Yes.*

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[4] We point out that the set $\mathcal{F}$ may be also an empty set.

[5] This is compatible with rule (R4)

2) Applying (R2) we obtain a new question, i.e the sequent $\bot, q \vdash \bot$, and now applying (R1) we obtain the sequent $\vdash \top$ so the answer is: *Yes.*
3) Applying (R2) we obtain the sequent $\vdash \bot$ so the asnwer is: *No.*
4) By (R2) we obtain the sequent $q \vdash \bot$ and after that by (R3) we obtain the sequent $\vdash \neg q$. Finally, by (R2) we obtain the sequent $\vdash \bot$ such that the answer is : *No.*
5) By (R2) we obtain the sequent $\neg q, q \vdash \bot$. Now by (R3) applied to the literal $\neg q$ we obtain the sequent $q \vdash q$, further by (R2) we obtain the sequent $\bot \vdash \bot$ and finally by (R1) we obtain the sequent $\vdash \top$ so the answer is : *Yes.*
6) Now by (R3) applied to clause p we obtain the sequent

$\neg p \vee q \vee \neg r, p \vee \neg q \vee s, p \vee s \vee \neg t \vdash \neg p$

By (R2) (and (R4) applied twice) we obtain the sequent

$q \vee \neg r \vdash \bot$

At this step applying (R3) we obtain two new sequents, i.e. questions
$\vdash \neg q$? and $\vdash r$?
The answer to the first question is *No*, so the final answer is also: *No.*

Concerning rules (R1)-(R4) we have this lemma.

Lemma 3. (Soundness of rules (R1)-(R4)). *Let $\mathcal{F}$ be any set of clauses. Suppose that we start with a sequent $\mathcal{F} \vdash \psi$, where ψ is a literal or the symbol $\bot$. If using rules* (R1)-(R4) *we obtain the sequent $\vdash \top$ or the sequent $\vdash \bot$, then ψ is / is not a logical consequence of set $\mathcal{F}$, respectively.*

Proof follows immediately from the fact that rules (R1)-(R4) are based on logical equivalences (6)-(9).

Let now $\mathcal{F} \vdash \psi$ be any sequent. By *Val($\mathcal{F} \vdash \psi$)* we denote its *truth value*, defined by:

> If ψ is a logical consequence of set $\mathcal{F}$ then *Val($\mathcal{F} \vdash \psi$)* is *true* otherwise *Val($\mathcal{F} \vdash \psi$)* is *false*.

According to this definition and to rules (R1)-(R4), i.e. to equivalences (6)-(9) we have the following equalities

(10) $Val(\vdash \top) = true$

$Val(\vdash \bot) = false$

$Val(\mathcal{F}, \bot \vdash \bot) = true$

$Val(\mathcal{F}, \top \vdash \psi) = Val(\mathcal{F} \vdash \psi)$

$Val(\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p) = Val(\mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot)$

$$Val(\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash \neg p) = Val(\mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \vdash \bot)$$

($\phi_i(p)$ is any clause containing p)

$$Val(\mathcal{F}, p_1 \vee \ldots \vee p_k \vdash \bot)$$

$$= Val(\mathcal{F} \vdash \neg p_1) \text{ and } \ldots \text{ and } Val(\mathcal{F} \vdash \neg p_k)$$

(where p_i is any literal, i.e. an atom or the negation of an atom)

Suppose that $\mathcal{F}$ is a finite set. Then, in fact, these equalities define the function Val recursively on the number of all member of set $\mathcal{F}$. Consequently, these equalities suggest how to calculate $Val(\mathcal{F} \vdash \psi)$. In other words we have the following assertion:

(11) *If $\mathcal{F}$ is a finite set then one can effectively calculate $Val(\vdash \psi)$, i.e. establish whether or not ψ is a logical consequence of set $\mathcal{F}$.*

Next we will prove the following basic theorem.

Theorem 1. (Completeness) *Let $\mathcal{F}$ be a set of some clauses and ψ a literal or the symbol $\bot$. Then:*

ψ is a logical consequence of set $\mathcal{F}$ if and only if starting with $\mathcal{F} \vdash \psi$ and applying rules (R1)-(R4) a finite number of times one can obtain the sequent $\vdash \top$.

Proof. The *if* - part follows immediately from Lemma 3. To prove the *only if* - part suppose now that ψ is a logical consequence of set $\mathcal{F}$. Then ψ is a logical consequence of some f i n i t e subset $\mathcal{A}$ of set $\mathcal{F}$ (for: every formal proof is finite). Next, by (11) we conclude that starting with the sequent $\mathcal{A} \vdash \psi$ and applying rules (R1)-(R4) a finite number of times one can obtain the sequent $\vdash \top$. Consequently, also starting with the sequent $\mathcal{F} \vdash \psi$ and applying rules (R1)-(R4) a finite number of times one can obtain the sequent $\vdash \top$. The proof is complete.

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