PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 19 (33), 1975, pp. 131—138

EQUATIONAL REFORMULATION OF FORMAL THEORIES

Slaviša B. Prešić

(Communicated March 28, 1975)

1. There are many important instances of formal theories (cf., for example, [4]), as propositional calculi, predicate calculi, formal arithmetics, axiomatic set theories and so on. Within the theory of universal algebras the concept of variety is of particular interest (cf. for example [2]). Every variety, with appropriate precision introduced, becomes a formal theory. Formal theories of this kind contain formulae of the form $t_1 = t_2$, where t_1 and t_2 are terms (constructed out of some primitive symbols, constants, individual variables and operation symbols; cf. [2]). Axioms are some formulae given in advance, as formulae of the form $t = t$, where t is any term. The rules of inference are (in agreement with elementary properties of equality):

$$(1) \qquad \frac{t_1 = t_2}{t_2 = t_1} \quad \frac{t_1 = t_2,\ t_2 = t_3}{t_1 = t_3} \quad \frac{t_1 = t_1',\ \ldots,\ t_n = t_n'}{\omega\, t_1 \ldots t_n = \omega\, t_1' \ldots t_n'}$$

(where t_i is any term and ω an operation symbol of length n).

A formal theory of this kind we shall call *equational formal theory*. One of our aims is to investigate the connection between equational and other formal theories.

2. Let $\mathcal{T}$ be a formal theory with axioms A_i ($i \in I$; I is a given set of indexes). By $\mathcal{T}(\sim)$ we denote the equational theory defined as follows:

1° The formulae of $\mathcal{T}$ play the role of individual variables of $\mathcal{T}(\sim)$; the symbol $\&$ is an operation symbol[1)] of length 2.

2° The axioms of $\mathcal{T}(\sim)$ are formulae of the form

(2) (a) $A_i \sim \top$ ($\top$ is an arbitrarily chosen axiom of $\mathcal{T}$; $i \in I$),

(b) $A \sim A$, $\& AB \sim \& BA$, $\&\& ABC \sim \& A \& BC$ and $\& A \top \sim A$ (A, B and C are terms of $\mathcal{T}(\sim)$).

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1) The terms of $\mathcal{T}(\sim)$ satisfy the following definition: (i) formulae of $\mathcal{T}$ are terms of $\mathcal{T}(\sim)$; (ii) if A and B are terms, then $\& AB$ is a term; (iii) every term is obtained by a finite number of applications of (i) and (ii).

The formulae of $\mathcal{T}(\sim)$ are of the form $A \sim B$, where A and B are terms and $\sim$ and $\&$ are not among the symbols of $\mathcal{T}$.

(c) Let

$$\frac{\Phi_1, \ldots, \Phi_k}{\Phi}$$

be any rule on inference of $\mathcal{T}$; then the formula

$$\& \ldots \& \Phi_1 \ldots \Phi_k \sim \& \& \ldots \& \Phi_1 \ldots \Phi_k \Phi$$

is an axiom of $\mathcal{T}(\sim)$.

3° The rule of inference of $\mathcal{T}(\sim)$ are

$$\frac{A \sim B}{B \sim A}, \quad \frac{A \sim B,\ B \sim C}{A \sim C}, \quad \frac{A_1 \sim B_1,\ A_2 \sim B_2}{\& A_1 A_2 \sim \& B_1 B_2}$$

(A, $B, \ldots$ are terms of $\mathcal{G}(\sim)$).

Note. In the sequel we shall write $A \& B$, $A \& B \& C$ etc., instead of $\& AB$, $\& A \& BC$ etc., respectively. The axiom (b) prevents us from possible confusion. For example, in this case axiom (c) becomes: $\Phi_1 \& \ldots \& \Phi_k \sim \Phi_1 \& \ldots \& \Phi_k \& \Phi$.

As we shall see soon, the symbol $\&$ is related to the metatheoretic *and* while the symbol $\sim$ is, so to say, a formalization of the relation that we call *equiconsequence*. In fact, we prove

Theorem 1. Let $P_1, \ldots, P_r$, $Q_1, \ldots, Q_s$ be formulae of $\mathcal{G}$; then $\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim Q_1 \& \ldots \& Q_s$ iff $P_1, \ldots, P_r \underset{\mathcal{T}}{\vdash} Q_1, \ldots, Q_s$ and

$$Q_1, \ldots, Q_s \underset{\mathcal{T}}{\vdash} P_1, \ldots, P_r.$$

We shall prove two lemmata first.

Lemma 1. If $P_1, \ldots, P_r \underset{\mathcal{T}}{\vdash} Q$, then $\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& Q$ where P_i and Q are formulae of $\mathcal{T}$.

Proof. We use induction on the length n of the shortest proof of $P_1, \ldots, P_r \underset{\mathcal{T}}{\vdash} Q$.

Case $n = 1$. Q is either P_i (for some $1 \leqslant i \leqslant r$) or A_j (for some $j \in I$). In both cases we have

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& Q$$

(for we have

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& P_i$$

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& \top, \quad \underset{\mathcal{T}(\sim)}{\vdash\!\!-} A_j \sim \top).$$

Case $n > 1$. The following subcases are possible:

(i) Q is P_i (for some $1 \leqslant i \leqslant r$);

(ii) Q is A_j (for some $j \in I$);

(iii) Q is a consequence of some preceding formulae by a rule

$$\frac{\Phi_1, \dots, \Phi_k}{\Phi}$$

In both (i) and (ii) we proceed as in Case $n=1$. In (iii) by induction hypothesis we have

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, \Phi_1$$

$$\vdots$$

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, \Phi_k.$$

Therefrom we derive immediately

(3) $$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, \Phi_1 \,\&\, \dots \,\&\, \Phi_k$$

i.e.,

(4) $$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \, P_r \,\&\, \Phi_1 \,\&\, \dots \,\&\, \Phi_k \,\&\, \Phi$$

[for $$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} \Phi_1 \,\&\, \dots \,\&\, \Phi_k \sim \Phi_1 \,\&\, \dots \,\&\, \Phi_k \,\&\, \Phi]$$

From (3) and (4) we derive

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, \Phi$$

i.e.

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, Q \quad \text{(for } Q \text{ is } \Phi).$$

Lemma 2. If $P_1, \dots, P_r \underset{\mathcal{T}}{\vdash} Q_1, \dots, Q_s$, then

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, Q_1 \,\&\, \dots \,\&\, Q_s.$$

This lemma is an immediate consequence of *Lemma* 1. Indeed, from $P_1, \dots, P_r \underset{\mathcal{T}}{\vdash} Q_1, \dots, Q_s$, by *Lemma* 1, it follows that

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, Q_1$$

$$\vdots$$

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, Q_s$$

are theorems; hence, $\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \,\&\, \dots \,\&\, P_r \sim P_1 \,\&\, \dots \,\&\, P_r \,\&\, Q_1 \,\&\, \dots \,\&\, Q_s$.

Now, we shall prove *Theorem* 1.

The "if" part. Suppose that $P_1, \ldots, P_r \underset{\mathcal{T}}{\vdash} Q_1, \ldots, Q_s$ and $Q_1, \ldots, Q_s \underset{\mathcal{T}}{\vdash} P_1, \ldots, P_r$. Therefrom, by *Lemma* 2,

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& Q_1 \& \ldots \& Q_s$$

and

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} Q_1 \& \ldots \& Q_s \sim Q_1 \& \ldots \& Q_s \& P_1 \& \ldots \& P_r$$

and hence

$$\underset{\mathcal{T}(\sim)}{\vdash\!\!-} P_1 \& \ldots \& P_r \sim Q_1 \& \ldots \& Q_s .$$

The „only if" part. Suppose that $\vdash P_1 \& \ldots \& P_r \sim Q_1 \& \ldots \& Q_s$ and let $A_1, \ldots, A_p, B_1, \ldots, B_q$ be arbitrary formulae of $\mathcal{T}$. Let us associate the sequents

$$A_1, \ldots, A_p \underset{\mathcal{T}}{\vdash} B_1, \ldots, B_q \text{ and } B_1, \ldots, B_q \underset{\mathcal{T}}{\vdash} A_1, \ldots, A_p$$

to the formula

$$A_1 \& \ldots \& A_p \sim B_1 \& \ldots \& B_q$$

and let Ψ denote this association.

Applying the mapping Ψ to the axioms of $\mathcal{T}(\sim)$ we obtain proofs from hypotheses in $\mathcal{T}$. For example, such proofs from hypotheses are $A_i \underset{\mathcal{T}}{\vdash} \top$, $\top \underset{\mathcal{T}}{\vdash} A_i$; $A, \top \underset{\mathcal{T}}{\vdash} A$; $\Phi_1, \ldots, \Phi_k \underset{\mathcal{T}}{\vdash} \Phi_1, \ldots, \Phi_k, \Phi$ and so on.

Moreover, the mapping Ψ is in accordance with rules of $\mathcal{T}(\sim)$ — in fact, the rules of $\mathcal{T}(\sim)$ are translated into true statements about proofs from hypotheses in $\mathcal{T}$. For example, to the rule

$$\frac{A \sim B,\ B \sim C}{A \sim C}$$

there corresponds the statement

$$\text{If } A \underset{\mathcal{T}}{\vdash} B,\ B \underset{\mathcal{T}}{\vdash} A,\ B \underset{\mathcal{T}}{\vdash} C,\ C \underset{\mathcal{T}}{\vdash} B, \text{ then } A \underset{\mathcal{T}}{\vdash} C,\ C \underset{\mathcal{T}}{\vdash} A.$$

In accordance with consideration, if we apply Ψ to the supposed theorem

$$P_1 \& \ldots \& P_r \sim Q_1 \& \ldots \& Q_s$$

we obtain proofs from hypotheses

$$P_1, \ldots, P_r \underset{\mathcal{T}}{\vdash} Q_1, \ldots, Q_s \text{ and } Q_1, \ldots, Q_s \underset{\mathcal{T}}{\vdash} P_1, \ldots, P_r.$$

This completes the proof of the theorem.

According to *Theorem* 1, just proved, we can say that in a sense $\mathcal{T}(\sim)$ is a formalization of deduction relation of $\mathcal{T}$. In particular, by *Theorem* 1 it follows that

$$A \underset{\mathcal{T}}{\vdash} B,\ B \underset{\mathcal{T}}{\vdash} A \text{ iff } \underset{\mathcal{T}(\sim)}{\vdash\!\!-} A \sim B.$$

3. By the next theorem a connection is established between the theorems of $\mathcal{T}$ and some theorems of $\mathcal{T}(\sim)$.

Lemma 3. Let A be any formula of $\mathcal{T}$; then

$$\underset{\mathcal{T}}{\vdash} A \quad \text{iff} \quad \underset{\mathcal{T}(\sim)}{\vdash\!\!-} A \sim \top$$

Proof. $\vdash A$ iff $A \vdash \top$, $\top \vdash A$ (by definition of $\vdash$)

iff $\vdash A \sim \top$ (by *Theorem* 1)

Hence, $\vdash A$ iff $\vdash A \sim \top$.

Let f denote a mapping of the set *For* ($\mathcal{T}$) (the set of formulae of $\mathcal{T}$) into the set *For* ($\mathcal{T}(\sim)$), defined by equality

$$f(A) \overset{\text{def}}{=} A \sim \top.$$

According to *Lemma* 3, by the injective mapping f *the set of theorems of* $\mathcal{T}$ *is mapped into the set of theorems of* $\mathcal{T}(\sim)$. Moreover, the mapping "translates" the proofs of $\mathcal{T}$ into (incomplete, but completable) proofs of $\mathcal{T}(\sim)$. In fact:

(i) if A_i is an axiom of $\mathcal{T}$, then $f(A_i)$, i.e. $A_i \sim \top$ is a theorem of $\mathcal{T}$;

(ii) if

$$\frac{\Phi_1, \ldots, \Phi_k}{\Phi}$$

is a rule of $\mathcal{T}$, then in $\mathcal{T}(\sim)$ it is the case that[2)]

$$f(\Phi_1), \ldots, f(\Phi_k) \vdash f(\Phi)$$

i.e.

$$\Phi_1 \sim \top, \ldots, \Phi_k \sim \top \vdash \Phi \sim \top$$

Having in mind the properties of the map f (it is $1-1$, it translates theorems and proofs of $\mathcal{T}$ into theorems and proofs of $\mathcal{T}(\sim)$) we can say:

$\mathcal{T}$ *is isomorphically embedded in* $\mathcal{T}(\sim)$ *by the mapping* f.

In this way we conclude that the following theorem is valid.

Theorem 2. Any formal theory can be isomorphically embedded in an equational formal theory.

4. Let $\mathcal{T}$ be a formulation of the classical propositional calculus, say P_2 of [1]. The axioms (inessentially modified) are formulas of the form[3)]

$$A \Rightarrow (B \Rightarrow A), \ (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)), \ (\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$$

(A, B, C are propositional formulas),

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2) Indeed, let $\Phi_1 \sim \top, \ldots, \Phi_k \sim \top$ be hypotheses. Using them we obtain $\Phi_1 \& \ldots \& \Phi_k \sim \top \& \ldots \& \top$, i.e. $\Phi_1 \& \ldots \& \Phi_k \sim \top$. But we have $\Phi_1 \& \ldots \& \Phi_k \sim \Phi_1 \& \ldots \& \Phi_k \& \Phi$ and hence $\Phi_1 \& \ldots \& \Phi_k \& \Phi \sim \top$. Therefore, $\top \& \Phi \sim \top$ and finally, $\Phi \sim \top$.

3) The primitive connectives are $\Rightarrow$ and $\neg$. The connectives $\wedge$, $\vee$ and $\Leftrightarrow$ are defined in terms of the primitive ones: for example, $A \vee B$ stand for $(A \Rightarrow B) \Rightarrow B$, etc.

The only rule is modus ponens:

$$\frac{A,\ A\Rightarrow B}{B}.$$

We prove

Theorem 3. Let $A_1, \ldots, A_p, B_1, \ldots, B_q$ be any formulas of P_2, then

$$\underset{P_2(\sim)}{\vdash\!\!-\!\!-} A_1 \& \ldots \& A_p \sim B_1 \& \ldots \& B_q \quad \textbf{iff} \quad \underset{P_2}{\vdash} A_1 \wedge \cdots \wedge A_p \Leftrightarrow B_1 \wedge \cdots \wedge B_q.$$

Proof. $\underset{P_2(\sim)}{\vdash\!\!-\!\!-} A_1 \& \ldots \& A_p \sim B_1 \& \ldots \& B_q$ **iff**

$$A_1, \ldots, A_p \underset{P_2}{\vdash} B_1. \ldots, B_q \quad \text{and} \quad B_1, \ldots, B_q \underset{P_2}{\vdash} A_1, \ldots, A_p$$

(by *Theorem* 1); but this is the case **iff**

$$A_1 \wedge \cdots \wedge A_p \underset{P_2}{\vdash} B_1 \wedge \cdots \wedge B_q \quad \text{and} \quad B_1 \wedge \cdots \wedge B_q \underset{P_2}{\vdash} A_1 \wedge \cdots \wedge A_p$$

(this is provable in P_2); again, this is the case **iff**

$$\underset{P_2}{\vdash} A_1 \wedge \cdots \wedge A_p \Rightarrow B_1 \wedge \cdots \wedge B_q \quad \text{and} \quad \underset{P_2}{\vdash} B_1 \wedge \cdots \wedge B_q \Rightarrow A_1 \wedge \cdots \wedge A_p$$

(by *deduction theorem*); by definition of $\Leftrightarrow$, this is the case **iff**

$$\underset{P_2}{\vdash} A_1 \wedge \cdots \wedge A_p \Leftrightarrow B_1 \wedge \cdots \wedge B_q.$$

Let us note that the preceding proof relies on the fact that in $\mathcal{T}$ viz. P_2 the following conditions are satisfied:

Condition 1. There is an operation in $\mathcal{T}$, in symbols $\wedge$, such that $A, B \underset{\mathcal{T}}{\vdash} A \wedge B$ and $A \wedge B \underset{\mathcal{T}}{\vdash} A, B$ (A, B are formulas of $\mathcal{T}$).

Condition 2. There is an operation in $\mathcal{T}$, in symbols $\Rightarrow$, such that $A \underset{\mathcal{T}}{\vdash} B$ iff $\vdash A \Rightarrow B$ (A, B are formulas of $\mathcal{T}$).

According to *Theorem* 3. to any theorem

$$A_1 \& \ldots \& A_p \sim B_1 \& \ldots \& B_q$$

of $P_2(\sim)$ there corresponds the theorem

$$A_1 \wedge \cdots \wedge A_p \Leftrightarrow B_1 \wedge \cdots \wedge B_q$$

of P_2. In other words, by substituting $\wedge$ and $\Leftrightarrow$ for & and $\sim$, respectively, the formulas of $P_2(\sim)$ are translated in to formulas of P_2, and, moreover, theorems are translated into theorems. Also, (this is proved easily), by this injective mapping the proofs of $P_2(\sim)$ are translated into (completable) proofs of P_2. On the other hand, the converse is also true in a sense; for example, to any theorem A of P_2 there corresponds (by *Lemma* 3) the theorem $A \sim \top$ of $P_2(\sim)$. Therefore, $P_2(\sim)$ is isomorphically embedded in P_2. The calculus $P_2(\sim)$ we shall also call an equational reformulation of P_2.

Remark. Let us note that the axioms of $P_2(\sim)$ can be transformed into axioms of Boolean algebra (cf. for example, [3], p. 5)

$$A \wedge \top \sim A \quad A \vee \neg \top \sim A$$

$$A \wedge \neg A \sim \neg \top, \quad A \vee \neg A \sim \top$$

$$(\mathcal{B}) \qquad A \wedge B \sim B \wedge A, \quad A \vee B \sim B \vee A$$

$$A \wedge (B \vee C) \sim (A \wedge B) \vee (A \wedge C), \quad A \vee (B \wedge C) \sim (A \vee B) \wedge (A \vee C)$$

and, in addition[4)]

$$(5) \qquad A \,\&\, B \sim A \wedge B$$

(A, B, C are any formulas of P_2; $\top$ is, say, $p \Rightarrow (p \Rightarrow p)$).

Proof. Using axioms and rules of $P_2(\sim)$, we prove easily $(\mathcal{B})$, (5), (6).

The formula (5) can be proved as follows. We have

$$A, B \underset{P_2}{\vdash} A \wedge B, \quad A \wedge B \underset{P_2}{\vdash} A, B$$

and, hence, by *Theorem* 1

$$\underset{P_2(\sim)}{\vdash\!\!-\!\!-} A \,\&\, B \sim A \wedge B.$$

Furthermore, the proof of, say

$$\frac{A \sim B}{\neg A \sim \neg B}$$

is as follows. Suppose that $\underset{P_2(\sim)}{\vdash\!\!-\!\!-} A \sim B$; then according to *Theorem* 3,

$$\underset{P_2}{\vdash} A \Leftrightarrow B.$$

Hence, using the well-known properties of P_2, we conclude that

$$\underset{P_2}{\vdash} \neg A \Leftrightarrow \neg B,$$

and hence, again by Theorem 3, we obtain $\underset{P_2(\sim)}{\vdash\!\!-\!\!-} \neg A \Leftrightarrow \neg B$.

Let us assume now that $(\mathcal{B})$, (5), and (6) hold and let us prove the axioms and rules of $P_2(\sim)$. Using $(\mathcal{B})$, (5), and (6) we can prove various facts about Boolean algebra, such as

$$\neg \neg A \sim A, \quad \neg (A \wedge B) \sim \neg A \vee \neg B, \quad A \Rightarrow B \sim \neg A \vee B \quad \text{etc.}$$

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4) Besides the axioms given above, we assume a number of properties of equality ($\sim$ stands for $=$):

$$(6) \qquad A \sim A, \quad \frac{A \sim B}{B \sim A}, \quad \frac{A \sim B, \; B \sim C}{A \sim C} \quad \frac{A \sim B}{\neg A \sim \neg B} \quad \frac{A \approx B, \; C \sim D}{A \wedge C \sim B \wedge D} \quad \frac{A \sim B, \; C \sim D}{A \vee C \sim B \vee D}$$

$$\frac{A \sim B, \; C \sim D}{A \,\&\, C \sim B \,\&\, D}$$

Using the last formula, we easily prove formulas

$$A \Rightarrow (B \Rightarrow A) \sim \top, \quad (A \Rightarrow (B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \sim \top,$$
$$(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A) \sim \top$$

i.e. a number of axioms of $P_2(\sim)$. These axioms are of the form (2) (a). The axioms of the form (2) (b) are proved easily, using (5). In a similar way we prove axioms of the form (3) (c), i.e. the formula $A \& (A \Rightarrow B) \sim A \& (A \Rightarrow B) \& B$.

Let $\mathcal{T}$ be a formal theory satisfying conditions 1. and 2. This means that the symbols $\wedge$, $\Rightarrow$ are either primitive in $\mathcal{T}$ or defined[5] such that we have 1. and 2. viz.

$$A, B \underset{\mathcal{T}}{\vdash} A \wedge B \quad \text{and} \quad A \wedge B \underset{\mathcal{T}}{\vdash} A, B$$

$$A \underset{\mathcal{T}}{\vdash} B \quad \textbf{iff} \quad \vdash A \Rightarrow B$$

Then we have the following theorem which is proved almost in the same way as in the case of P_2.

Theorem 4.

1° $\underset{\mathcal{T}}{\vdash} A$ **iff** $\underset{\mathcal{T}(\sim)}{\vdash\!\!-} A \sim \top$

2° $\underset{\mathcal{T}}{\vdash} A \Leftrightarrow B$ **iff** $\underset{\mathcal{T}(\sim)}{\vdash\!\!-} A \sim B.$

In other words, if the conditions 1. and 2. are satisfied, $\mathcal{T}(\sim)$ is an equational reformulation of $\mathcal{T}$.

Finally, let us note that there are various formal theories satisfying conditions 1. and 2. — for example, the classical propositional calculus, the intuitionistic propositional calculus and many others.

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[5] Then, for example, $A \wedge B$ stand for a formula constructed in some way out of subformulas of A and B.