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AN ALGORITHM FOR THE SOLUTION OF A 2×2 SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS

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Summary

This paper treats the problem of simultaneous determination of all solutions of the system of algebraic equations

$$J_1(x, y) \equiv A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0,$$

$$J_2(x, y) \equiv A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0,$$

on the assumption that they are different. The algorithm is based on the use of basic ideas of an iterative procedure for factorisation of polynomials given in papers [1] and [2]. This analysis would be preliminarily made and its analogy of treatment would be transferred to the construction of algorithm for the solution of given problem. For treatment of this kind of problem immediate cause was conditionned by practical needs.

Introduction and statement of the problem

In papers [1] and [2] an iterative procedure for simultaneous determination of all roots of an algebraic polynomial was given, under the assumption that they differ one from another.

For example, in the case of an algebraic equation

$$P(x) \equiv x^3 + p_2x^2 + p_1x + p_0 = 0, \tag{1}$$

whose roots are a, b, c, it was demonstrated that they represent limit values of a series a_n, b_n, c_n defined as follows

$$\begin{aligned} a_{n+1} &= a_n - \frac{P(a_n)}{(a_n-b_n)(a_n-c_n)}, \\ b_{n+1} &= b_n - \frac{P(b_n)}{(b_n-a_n)(b_n-c_n)}, \\ c_{n+1} &= c_n - \frac{P(c_n)}{(c_n-a_n)(c_n-b_n)}. \end{aligned} \tag{2}$$

For starting data a_0, b_0, c_0, approximative values of roots a, b, c are selected.

Otherwise, formulae (2) are derived from following polynomial identity

$$(3)\qquad \begin{aligned}&(x-a_{n+1})(x-b_n)(x-c_n)+(x-a_n)(x-b_{n+1})(x-c_n)+\\&+(x-a_n)(x-b_n)(x-c_{n+1})-2(x-a_n)(x-b_n)(x-c_n)=P(x),\end{aligned}$$

that may be written in this way

$$(4)\qquad \begin{aligned}&(x-a_n)(x-b_n)(x-c_n)-(a_{n+1}-a_n)(x-b_n)(x-c_n)-\\&-(b_{n+1}-b_n)(x-c_n)(x-a_n)-(c_{n+1}-c_n)(x-a_n)(x-b_n)=P(x).\end{aligned}$$

Now, let's observe the set of all polynomial expressions of a_n, b_n, c_n. Various constants, as well as a variable x, may take part here. For example such is the case for

$$a_n+b_n,\ a_nb_n+3c_n,\ x-a_n,\ (x-a_n)(x-b_n)(x-c_n), \text{ etc.}$$

In this set we define the operator δ in following way

$$(5)\qquad \begin{aligned}&\delta a_n \overset{\text{def}}{=} a_{n+1}-a_n,\ \delta b_n \overset{\text{def}}{=} b_{n+1}-b_n,\ \delta c_n \overset{\text{def}}{=} c_{n+1}-c_n,\\&\delta u \overset{\text{def}}{=} 0, \ (u \text{ is an expression which does not include index } n)\\&\delta(u+v) \overset{\text{def}}{=} \delta u+\delta v,\ \delta(uv) \overset{\text{def}}{=} v\delta u+u\delta v \ \ (u \text{ and } v \text{ are the expressions}\end{aligned}$$

whatsoever); it means that for the operator δ similar formulae are valable as for the differentiation. For example:

$$\begin{aligned}&\delta(a_n+b_n)=\delta a_n+\delta b_n=a_{n+1}-a_n+b_{n+1}-b_n,\\&\delta(x-a_n)=\delta x-\delta a_n=0-(a_{n+1}-a_n)=a_n-a_{n+1},\\&\delta[(x-a_n)(x-b_n)(x-c_n)]=(x-b_n)(x-c_n)\,\delta(x-a_n)+\\&+(x-a_n)(x-c_n)\,\delta(x-b_n)+(x-a_n)(x-b_n)\,\delta(x-c_n)=\\&=(x-b_n)(x-c_n)(a_n-a_{n+1})+(x-a_n)(x-c_n)(b_n-b_{n+1})+\\&+(x-a_n)(x-b_n)(c_n-c_{n+1}).\end{aligned}$$

On the basis of last equality (from above mentionned examples) the formula (4) may be written in this way

$$(6)\qquad (x-a_n)(x-b_n)(x-c_n)+\delta[(x-a_n)(x-b_n)(x-c_n)]-P(x)=0.$$

Introducing the designation

$$Q_n=(x-a_n)(x-b_n)(x-c_n)-P(x),$$

the equality (6) may be presented in the from

$$(7)\qquad Q_n+\delta Q_n=0,$$

as we have

$$\begin{aligned}Q_n+\delta Q_n&=(x-a_n)(x-b_n)(x-c_n)-P(x)+\delta[(x-a_n)(x-b_n)(x-c_n)-\\&-P(x)]=(x-a_n)(x-b_n)(x-c_n)-P(x)+\\&+\delta[(x-a_n)(x-b_n)(x-c_n)]-\delta P(x)=(x-a_n)(x-b_n)(x-c_n)+\\&+\delta[(x-a_n)(x-b_n)(x-c_n)]-P(x), \quad (\delta P(x)=0).\end{aligned}$$

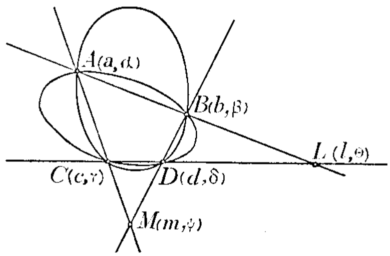
Basic method

Let's now consider the system of equations

(8)
$$J_1(x, y) \equiv A_1x^2+2B_1xy+C_1y^2+2D_1x+2E_1y+F_1=0,$$
$$J_2(x, y) \equiv A_2x^2+2B_2xy+C_2y^2+2D_2x+2E_2y+F_2=0,$$

supposing that its solutions are (a, α), (b, β), (c, γ), (d, δ).

Following schema for further treatment may be useful



The system of equations (8) is equivalent to the system of equations

(9)
$$AB\cdot CD=0,$$
$$AC\cdot BD=0,$$

where $AB, \ldots, BD$ denote following expressions

$$AB \overset{\text{def}}{=} (\beta-\alpha)(x-a)-(b-a)(y-\alpha),$$

— — — — — — — — — —

$$BD \overset{\text{def}}{=} (\delta-\beta)(x-b)-(d-b)(y-\beta).$$

Naturally, $AB=0, \ldots, BD=0$ are the equations of straight lines through the points A and $B, \ldots,$ or B and D respectively.

Equations of the system (9) are certain linear combinations equation of system (8); in other words there exists certain constantes λ, μ, ρ, φ so that following equalities are valable

(10)
$$AB\cdot CD+\lambda J_1(x, y)+\mu J_2(x, y)=0,$$
$$AC\cdot BD+\rho J_1(x, y)+\varphi J_2(x, y)=0.$$

Basic idea of algorithm given in this paper is this: iterative procedure of the system of equations (8) in the limit shall result in the form (10). Pursuant this point, preliminarily we introduce the designations

$$R_n=A_nB_n\cdot C_nD_n+\lambda_nJ_1(x, y)+\mu_nJ_2(x, y),$$
$$S_n=A_nC_n\cdot B_nD_n+\rho_nJ_1(x, y)+\varphi_nJ_2(x, y).$$

wheie $A_nB_n, \ldots, B_nD_n$ are following expressions

$$A_nB_n = (\beta_n - \alpha_n)(x - a_n) - (b_n - a_n)(y - \alpha_n),$$

$$C_nD_n = (\delta_n - \gamma_n)(x - c_n) - (d_n - c_n)(y - \gamma_n),$$

$$A_nC_n = (\gamma_n - \alpha_n)(x - a_n) - (c_n - a_n)(y - \alpha_n),$$

$$B_nD_n = (\delta_n - \beta_n)(x - b_n) - (d_n - b_n)(y - \beta_n).$$

In expression for R_n and S_n, 12 series are participating

$$a_n, \alpha_n, b_n, \beta_n, c_n, \gamma_n, d_n, \delta_n, \lambda_n, \mu_n, \rho_n, \varphi_n.$$

Our aim is to define these series so that

$$a_n \to a, \quad \alpha_n \to \alpha, \ldots, \quad \rho_n \to \rho, \quad \varphi_n \to \varphi,$$

when $n \to \infty$.

Conformably to equation (7) which refer to algebraic polynomials, fo1 R_n and S_n following conditions are posed

$$\text{(11)} \qquad \begin{aligned} R_n + \delta R_n &= 0, \\ S_n + \delta S_n &= 0. \end{aligned}$$

Now, operator δ refers to polynomial expressions for

$$a_n, \alpha_n, b_n, \beta_n, \ldots, \rho_n, \varphi_n$$

and is introduced with equalities similar to equalities (5).

On the basis of the definition of operator δ we have

$$\begin{aligned}
\delta R_n &= \delta [A_nB_n \cdot C_nD_n + \lambda_n J_1(x, y) + \mu_n J_2(x, y)] = C_nD_n \delta A_nB_n + \\
&+ A_nB_n \delta C_nD_n + J_1(x, y)\,\delta\lambda_n + J_2(x, y)\,\delta\mu_n = \\
&= [(\delta_n - \gamma_n)(x - c_n) - (d_n - c_n)(y - \gamma_n)] \cdot \delta [(\beta_n - \alpha_n)(x - a_n) - \\
&- (b_n - a_n)(y - \alpha_n)] + [(\beta_n - \alpha_n)(x - a_n) - \\
&- (b_n - a_n)(y - \alpha_n)] \cdot \delta [(\delta_n - \gamma_n)(x - c_n) - (d_n - c_n)(y - \gamma_n)] + \\
&+ J_1(x, y)\,\delta\lambda_n + J_2(x, y)\,\delta\mu_n = [(\beta_{n+1} - \beta_n - \alpha_{n+1} + \alpha_n)(x - a_n) - \\
&- (\beta_n - \alpha_n)(a_{n+1} - a_n) - (b_{n+1} - b_n) - (a_{n+1} + a_n)(y - \alpha_n) + \\
&+ (b_n - a_n)(\alpha_{n+1} - \alpha_n)] \cdot [(\delta_n - \gamma_n)(x - c_n) - (d_n - c_n)(y - \gamma_n)] + \\
&+ [(\delta_{n+1} - \delta_n - \gamma_{n+1}\gamma_n)(x - c_n) - (\delta_n - \gamma_n)(c_{n+1} - c_n) - \\
&- (d_{n+1} - d - c_{n+1} + c_n)(y - \gamma_n) + (d_n - c_n)(\gamma_{n+1} - \gamma_n)] \cdot [(\beta_n - \alpha_n)(x - a_n) - \\
&- (b_n - a_n)(y - \alpha_n)] + (\lambda_{n+1} - \lambda_n) J_1(x, y) + (\mu_{n+1} - \mu_n) J_2(x, y).
\end{aligned}$$

Similar procedure lead us to the expression for δS_n, too. Using derived expressions for δR_n and δS_n equalities (11) become

$$
\begin{aligned}
&[(\beta_n-\alpha_n)(x-a_n)-(b_n-a_n)(y-\alpha_n)]\cdot[(\delta_n-\gamma_n)(x-c_n)-(d_n-c_n)(y-\gamma_n)]+\\
&\quad+[(\beta_{n+1}-\beta_n-\alpha_{n+1}+\alpha_n)(x-a_n)-(\beta_n-\alpha_n)(a_{n+1}-a_n)-\\
&\quad-(b_{n+1}-b_n-a_{n+1}+a_n)(y-\alpha_n)+(b_n-a_n)(\alpha_{n+1}-\alpha_n)]\cdot[(\delta_n-\gamma_n)(x-c_n)-\\
&\quad-(d_n-c_n)(y-\gamma_n)]+[(\delta_{n+1}-\delta_n-\gamma_{n+1}+\gamma_n)(x-c_n)-\\
&\quad-(d_{n+1}-d_n-c_{n+1}+c_n)(y-\gamma_n)+(d_n-c_n)(\gamma_{n+1}-\gamma_n)]\cdot[(\beta_n-\alpha_n)(x-a_n)-\\
&\quad-(b_n-a_n)(y-\alpha_n)]+\lambda_{n+1}J_1(x,y)+\mu_{n+1}J_2(x,y)=0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&[(\gamma_n-\alpha_n)(x-a_n)-(c_n-a_n)(y-\alpha_n)]\cdot[(\delta_n-\beta_n)(x-b_n)-\\
&\quad-(d_n-b_n)(y-\beta_n)]+[(\gamma_{n+1}-\gamma_n-\alpha_{n+1}+\alpha_n)(x-a_n)-\\
&\quad-(\gamma_n-\alpha_n)(a_{n+1}-a_n)-(c_{n+1}-c_n-a_{n+1}+a_n)(y-\alpha_n)+\\
&\quad+(c_n-a_n)(\alpha_{n+1}-\alpha_n)]\cdot[(\delta_n-\beta_n)(x-b_n)-(d_n-b_n)(y-\beta_n)]+\\
&\quad+[(\delta_{n+1}-\delta_n-\beta_{n+1}+\beta_n)(x-b_n)-(\delta_n-\beta_n)(b_{n+1}-b_n)-\\
&\quad-(d_{n+1}-d_n-b_{n+1}+b_n)(y-\beta_n)+\\
&\quad+(d_n-b_n)(\beta_{n+1}-\beta_n)]\cdot[(\gamma_n-\beta_n)(x-a_n)-\\
&\quad-(c_n-a_n)(y-\alpha_n)]+\rho_{n+1}J_1(x,y)+\varphi_{n+1}J_2(x,y)=0.
\end{aligned}
\tag{13}
$$

Equalities (12) and (13) represent polynomial identities. These identities facilitate to determine

$$a_{n+1},\quad \alpha_{n+1},\quad b_{n+1},\quad \beta_{n+1},\ \dots,\ \lambda_{n+1},\quad \mu_{n+1},\quad \rho_{n+1},\quad \varphi_{n+1}$$

as the functions of

$$a_n,\quad \alpha_n,\quad b_n,\quad \beta_n,\quad c_n,\quad \gamma_n,\quad d_n,\quad \delta_n.$$

This determination shall be accomplished in such a way that in equalities (12) and (13) instead of (x, y) we replace (a_n, α_n), (b_n, β_n), (c_n, γ_n), (d_n, δ_n), after that, following eight equations are obtained

$$
(\alpha_n-\beta_n)a_{n+1}+(b_n-a_n)\alpha_{n+1}+\frac{J_1(a_n,\alpha_n)\lambda_{n+1}+J_2(a_n,\alpha_n)\mu_{n+1}}{(\delta_n-\gamma_n)(a_n-c_n)-(d_n-c_n)(\alpha_n-\gamma_n)}=a_n(\alpha_n-\beta_n)+\alpha_n(b_n-a_n),
\tag{14}
$$

$$
(\alpha_n-\gamma_n)a_{n+1}+(c_n-a_n)\alpha_{n+1}+\frac{J_1(a_n,\alpha_n)\rho_{n+1}+J_2(a_n,\alpha_n)\varphi_{n+1}}{(\delta_n-\beta_n)(a_n-b_n)-(d_n-b_n)(\alpha_n-\beta_n)}=a_n(\alpha_n-\gamma_n)+\alpha_n(c_n-a_n),
\tag{15}
$$

$$
(\alpha_n-\beta_n)b_{n+1}+(b_n-a_n)\beta_{n+1}+\frac{J_1(b_n,\beta_n)\lambda_{n+1}+J_2(b_n,\beta_n)\mu_{n+1}}{(\delta_n-\gamma_n)(b_n-c_n)-(d_n-c_n)(\beta_n-\gamma_n)}=b_n(\alpha_n-\beta_n)+\beta_n(b_n-a_n),
\tag{16}
$$

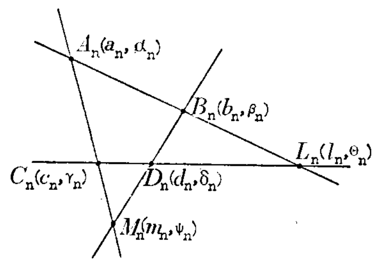
$$
(\beta_n-\delta_n)b_{n+1}+(d_n-b_n)\beta_{n+1}+\frac{J_1(b_n,\beta_n)\rho_{n+1}+J_2(b_n,\beta_n)\varphi_{n+1}}{(\gamma_n-\alpha_n)(b_n-a_n)-(c_n-a_n)(\beta_n-\alpha_n)}=b_n(\beta_n-\delta_n)+\beta_n(d_n-b_n),
\tag{17}
$$

$$(18)\qquad (\gamma_n-\delta_n)\,c_{n+1}+(d_n-c_n)\,\gamma_{n+1}+\frac{J_1(c_n,\gamma_n)\,\lambda_{n+1}+J_2(c_n,\gamma_n)\,\mu_{n+1}}{(\beta_n-\alpha_n)(c_n-a_n)-(b_n-a_n)(\gamma_n-\alpha_n)}=$$
$$=c_n(\gamma_n-\delta_n)+\gamma_n(d_n-c_n),$$

$$(19)\qquad (\alpha_n-\gamma_n)\,c_{n+1}+(c_n-a_n)\,\gamma_{n+1}+\frac{J_1(c_n,\gamma_n)\,\rho_{n+1}+J_2(c_n,\gamma_n)\,\varphi_{n+1}}{(\delta_n-\beta_n)(c_n-b_n)-(d_n-b_n)(\gamma_n-\beta_n)}=$$
$$=c_n(\alpha_n-\gamma_n)+\gamma_n(c_n-a_n),$$

$$(20)\qquad (\gamma_n-\delta_n)\,d_{n+1}+(d_n-c_n)\,\delta_{n+1}+\frac{J_1(d_n,\delta_n)\,\lambda_{n+1}+J_2(d_n,\delta_n)\,\mu_{n+1}}{(\beta_n-\alpha_n)(d_n-a_n)-(b_n-a_n)(\delta_n-\alpha_n)}=$$
$$=d_n(\gamma_n-\delta_n)+\delta_n(d_n-c_n),$$

$$(21)\qquad (\beta_n-\delta_n)\,d_{n+1}+(d_n-b_n)\,\delta_{n+1}+\frac{J_1(d_n,\delta_n)\,\rho_{n+1}+J_2(d_n,\delta_n)\,\varphi_{n+1}}{(\gamma_n-\alpha_n)(d_n-a_n)-(c_n-a_n)(\delta_n-\alpha_n)}=$$
$$=d_n(\beta_n-\delta_n)+\delta_n(d_n-b_n).$$



If we designate, further, on the basis of given schema, with L_n and M_n average points of straight lines A_nB_n and C_nD_n, and A_nC_n and B_nD_n respectively, coordinates of these points are given by formulae

$$l_n=\frac{(d_n-c_n)\,[(\alpha_n-\gamma_n)(b_n-a_n)-a_n(\beta_n-\alpha_n)]+c_n(\delta_n-\gamma_n)(b_n-a_n)}{(b_n-a_n)(\delta_n-\gamma_n)-(d_n-c_n)(\beta_n-\alpha_n)},$$

$$\theta_n=\frac{(\beta_n-\alpha_n)\,[(c_n(\delta_n-\gamma_n)-\gamma_n(d_n-c_n)]+(\delta_n-\gamma_n)\,[\alpha_n(b_n-a_n)-a_n(\beta_n-\alpha_n)]}{(b_n-a_n)(\delta_n-\gamma_n)-(d_n-c_n)(\beta_n-\alpha_n)},$$

$$m_n=\frac{(d_n-b_n)\,[(\alpha_n-\beta_n)(c_n-a_n)-a_n(\gamma_n-\alpha_n)]+b_n(\delta_n-\beta_n)(c_n-a_n)}{(c_n-a_n)(\delta_n-\beta_n)-(d_n-b_n)(\gamma_n-\alpha_n)},$$

$$\psi_n=\frac{(\gamma_n-\alpha_n)\,[b_n(\delta_n-\beta_n)-\beta_n(d_n-b_n)]+(\delta_n-\beta_n)\,[\alpha_n(c_n-a_n)-a_n(\gamma_n-\alpha_n)]}{(c_n-a_n)(\delta_n-\beta_n)-(d_n-b_n)(\gamma_n-\alpha_n)}.$$

Substituting in basic formulae (12) and (13) values for (x, y) with calculated values (l_n, θ) and (m_n, ψ_n) we obtain similarly to equations (14)—(21), four equations more

$$(22)\qquad J_1(l_n,\theta_n)\,\lambda_{n+1}+J_2(l_n,\theta_n)\,\mu_{n+1}=0,$$

(because the point $L_n(\gamma_n, \theta)$ lays in the intersection of straight lines A_nB_n and C_nD_n).

$$\begin{aligned}
&[(\theta_n-\gamma_n)\,a_{n+1}+(c_n-l_n)\,\alpha_{n+1}+(\alpha_n-\theta_n)\,c_{n+1}+\\
&\quad+(l_n-a_n)\,\gamma_{n+1})]\cdot[(\delta_n-\beta_n)(l_n-b_n)-(d_n-b_n)(\theta_n-\beta_n)]+\\
&\quad+[(\theta_n-\delta_n)\,b_{n+1}+(d_n-l_n)\,\beta_{n+1}+(\beta_n-\theta_n)\,d_{n+1}+\\
&\quad+(l_n-b_n)\,\delta_{n+1}]\cdot[(\gamma_n-\alpha_n)(l_n-a_n)-(c_n-a_n)(\theta_n-\alpha_n)]+\\
(23)\qquad&\quad+J_1(l_n,\theta_n)\,\rho_{n+1}+J_2(l_n,\theta_n)\,\varphi_{n+1}=[(\gamma_n-\alpha_n)(l_n-a_n)-\\
&\quad-(c_n-a_n)(\theta_n-\alpha_n)]\cdot[(d_n-b_n)(\theta_n-\beta_n)-(\delta_n-\beta_n)(l_n-b_n)_1]+\\
&\quad+[a_n(\theta_n-\gamma_n)+\alpha_n(c_n-l_n)+c_n(\alpha_n-\theta_n)+\\
&\quad+\gamma_n(l_n-a_n)]\cdot[(\delta_n-\beta_n)(l_n-b_n)-(d_n-b_n)(\theta_n-\beta_n)]+\\
&\quad+[b_n(\theta_n-\delta_n)+\beta_n(d_n-l_n)+d_n(\beta_n-\theta_n)+\delta_n(l_n-b_n)]\times\\
&\quad\times[(\gamma_n-\alpha_n)(l_n-a_n)-(c_n-a_n)(\theta_n-\alpha_n)],
\end{aligned}$$

$$\begin{aligned}
&[(\psi_n-\beta_n)\,a_{n+1}+(b_n-m_n)\,\alpha_{n+1}+(\alpha_n-\psi_n)\,b_{n+1}+\\
&\quad+(m_n-a_n)\,\beta_{n+1}]\cdot[(\delta_n-\gamma_n)(m_n-c_n)-(d_n-c_n)(\psi_n-\gamma_n)]+\\
&\quad+[(\psi_n-\delta_n)\,c_{n+1}+(d_n-m_n)\,\gamma_{n+1}+(\gamma_n-\psi_n)\,d_{n+1}+\\
&\quad+(m_n-c_n)\,\delta_{n+1})]\cdot[(\beta_n-\alpha_n)(m_n-a_n)-(b_n-a_n)(\psi_n-\alpha_n)]+\\
(24)\qquad&\quad+J_1(m_n,\psi_n)\,\lambda_{n+1}+J_2(m_n,\psi_n)\,\mu_{n+1}=\\
&\quad=[(\beta_n-\alpha_n)(m_n-a_n)-(b_n-a_n)(\psi_n-\alpha_n)]\cdot[(d_n-c_n)(\psi_n-\gamma_n)-\\
&\quad-(\delta_n-\gamma_n)(m_n-c_n)]+[a_n(\psi_n-\beta_n)+\alpha_n(b_n-m_n+b_n(\alpha_n-\psi_n)+\\
&\quad+\beta_n(m_n-a_n)]\cdot[(\delta_n-\gamma_n)(m_n-c_n)-(d_n-c_n)(\psi_n-\gamma_n)]+\\
&\quad+[c_n(\psi_n-\delta_n)+\gamma_n(d_n-m_n)+d_n(\gamma_n-\psi_n)+\\
&\quad+\delta_n(m_n-c_n]\cdot[(\beta_n-\alpha_n)(m_n-a_n)-(b_n-a_n)(\psi_n-\alpha_n)],
\end{aligned}$$

$$(25)\qquad J_1(m_n,\psi_n)\,\rho_{n+1}+J_2(m_n,\psi_n)\,\varphi_{n+1}=0,$$

(because the point $M_n(m_n, \psi_n)$ lays in the intersection of straight lines A_nC_n and B_nD_n).

System of equations (14) — (25) represents a system of linear algebraic equations which aids the determination of series

$$a_{n+1},\ \alpha_{n+1},\ b_{n+1},\ \beta_{n+1},\ \ldots,\ \lambda_{n+1},\ \mu_{n+1},\ \rho_{n+1},\ \varphi_{n+1}$$

in function of series

$$a_n,\ \alpha_n,\ a_n,\ \beta_n,\ c_n,\ \gamma_n,\ d_n,\ \delta_n.$$

In relation to these series following assertion is valable:

If the series a_n, α_n, b_n, β_n, c_n, γ_n, d_n, δ_n, λ_n, μ_n, ρ_n, φ_n converges one after another towards a, α, b, β, c, γ, d, δ, λ, μ, ρ, φ and if $\lambda\varphi-\rho\mu\neq 0$, then the points $A(a,\alpha)$, $B(b,\beta)$, $C(c,\gamma)$, $D(d,\delta)$ determine all solutions of system (8).

Proof.

Equalities (12) and (13) in limit case give

$$[(\beta-\alpha)(x-a)-(b-a)(y-\alpha)]\cdot[(\delta-\gamma)(x-c)-(d-c)(y-\gamma)]+$$
$$+\lambda J_1(x, y)+\mu J_2(x, y)=0,$$
$$[(\gamma-\alpha)(x-a)-(c-a)(y-\alpha)]\cdot[(\delta-\beta)(x-b)-(d-b)(y-\beta)]+$$
$$+\rho J_1(x, y)+\varphi J_2(x, y)=0.$$

On the basis of previous equalities taking into account the assumption that $\lambda\varphi-\rho\mu\neq 0$, it may be concluded that the system $J_1(x, y)=0$, $J_2(x, y)=0$ is equivalent to the system

$$[(\beta-\alpha)(x-a)-(b-a)(y-\alpha)]\cdot[(\delta-\gamma)(x-c)-(d-c)(y-\gamma)]=0,$$
$$[(\gamma-\alpha)(x-a)-(c-a)(y-\alpha)]\cdot[(\delta-\beta)(x-b)-(d-b)(y-\beta)]=0.$$

The proof is finished, as all solutions of that system are the points $A(a, \alpha)$, $B(b, \beta)$, $C(c, \gamma)$, $D(d, \delta)$.

Note. On the basis of investigations with are not completely finished it may be assumed that the convergence is *quadratic.*

Application

Progressivity of application of exposed algorithm may be seen from the following: On the basis of given starting data

$$(a_0, \alpha_0),\ (b_0, \beta_0),\ (c_0, \gamma_0),\ (d_0, \delta_0)$$

by use of formula for l, θ, m, ψ, we first calculate the values (l_0, θ_0), (m_0, ψ_0). In this way a complete group of data is formed, necessary for calculation — on the basis of the system of linear algebraic equations (14)—(25) of values (a_1, α_1), (b_1, β_1), (c_1, γ_1), (d_1, δ_1), λ_1, μ_1, ρ_1, φ_1 and they, after calculations by using formulae for l, θ, m, ψ of values (l_1, θ), (m_1, θ_1), represent first corrections of initial conditions. Further procedure continue in the same way, by calculating (a_2, α_2), (b_2, β_2), (c_2, γ_2), (d_2, δ_2), λ_2, μ_2, ρ_2, φ_2, until results of desired precisions are obtained.

This procedure will be illustrated by following example.

A system of equations is given

$$J_1(x, y)\equiv x^2-4xy+2y^2-x-2y=0,$$
$$J_2(x, y)\equiv 3x^2-14xy+2y^2-3x+8y=0,$$

its solutions are known

$$A(a; \alpha)=A(1; 3), \quad B(b; \beta)=B(5; 1),$$
$$C(c; \gamma)=C(0; 0), \quad D(d; \delta)=D(1; 0).$$

By the application of exposed algorithm it is necessary to determine solutions of the given system, under assumption that they are unkown, starting with following initial conditions

$$A_0(a_0; \alpha_0)=A_0(2; 4), \qquad B_0(b_0; \beta_0)=B_0(3; 2),$$
$$C_0(c_0; \gamma_0)=C_0(-1; -1), \qquad D_0(d_0; \delta_0)=D_0(0{,}5; -1).$$

unknovwn \ i-th step	0	1	2	3	4	5	6	7	8
a_i	2	1,206949	0,875619	1,005097	1,000024	1,000000	1,000000	0,999999	1,000000
α_i	4	3,070373	3,071830	3,000608	2,999955	3,000000	3,000000	3, 00000	3,000000
b_i	3	3,056078	5,030799	5,037009	4,999879	5,000013	5,000012	5,000001	5,000001
β_i	2	1,337529	1,105335	1,010320	0,999981	1,000002	1,000003	1,000001	1,000001
c_i	−1	−0,375857	−0,021557	0,007307	0,000019	−0,000000	0,000000	−0,000000	0,000000
γ_i	−1	−0,122115	0,097986	0,002978	0,000009	−0,000000	−0,000000	−0,000000	−0,000000
d_i	0,5	0,964449	0,993457	0,999251	0,999987	1,000000	1,000000	1,000000	1,000000
δ_i	−1	−0,033672	0,029587	0,001278	0,000000	0,000000	−0,000000	0,000000	−0,000000
λ_i	—	−2,048944	−3,676849	−2,994956	−2,999973	−3,000009	−3,000008	−3,000003	−3,000001
μ_i	—	0,741944	1,192996	0,996661	0,999979	1,000003	1,000002	1,000001	1,000000
ρ_i	—	−1,332192	−2,149933	−1,508434	−1,500044	−1,500004	−1,500004	−1,000001	−1,500001
φ_i	—	−0,033785	−0,328192	−0,505963	−0,499956	−0,500001	−0,500002	−0,500001	−0,500001

Results of calculations obtained on digital computer CII 10070 are given in following table.

As may be seen from the data indicated in the table using these iterative approximations, solutions with precision on fifth digit are obtained, with total working time of CII 10070 computer of 1,79 min.

The same example was solved with other initial conditions

$$A_0(a_0; \alpha_0) = A_0(0; 12), \qquad B_0(b_0, \beta_0) = B_0(3; 10)$$

$$C_0(c_0; \gamma_0) = C_0(-4; -2), \qquad D_0(d_0; \delta_0) = D_0(0{,}1; -3)$$

and the same solutions were obtained after twelve iterations, total computer working time being 1,92 min.

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