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# FIN-SET: A SYNTACTICAL DEFINITION OF FINITE SETS

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Abstract. We state Fin-set, by which one founds the notion of finite sets in a syntactical way. Any finite set $\{a_1, a_2, \dots, a_n\}$ is defined as a well formed term of the form $S(a_1 + (a_2 + (\cdots + (a_{n-1} + a_n)\cdots)))$, where $+$ is a binary and $S$ a unary operational symbol. Related to the operational symbol $+$ the term-substitutions (1) are introduced. Definition of finite sets is called syntactical because by two algorithms Set-alg and Calc one can effectively establish whether any given set-terms are equal or not equal.

All other notions related to finite sets, like $\in$, ordered pair, Cartesian product, relation, function, cardinal number are defined as terms as well. Each of these definitions is recursive. For instance, $\in$ is defined by

$x \in S(a_1)$ iff $x = a_1$

$x \in S(a_1 + \cdots + a_n)$ iff $x = a_1$ or $x \in S(a_2 + \cdots + a_n)$

$x \notin \emptyset$ ($\emptyset$ denotes the empty set)

## 1. The key idea

Finite sets are usually expressed by some set-terms like $\{a\}$, $\{a, \{b, c\}\}$, $\{a_1, \dots, a_n\}$. Related to such terms there are infinite number of 'algebraic laws' like

$$(*) \qquad \{x, x\} = \{x\}, \quad \{x, y\} = \{y, x\}, \quad \{x, y, z\} = \{z, x, y\}$$

which express various properties of finite sets. In order to state all such algebraic laws we use the following idea: We 'divide' the notion of finite set into two 'parts'. The first one is the *preset*, formalized by means of a binary operational symbol $+$. The second one is the *set-maker*, formalized by an unary operational symbol $S$.

For instance, using $S$ and $+$ the ordinary set-terms $\{a\}$, $\{\{a, b\}, c\}$, $\{a_1, \dots, a_n\}$ are represented by the following $(S, +)$-terms

$$S(a), \quad S(S(a + b) + c), \quad S(a_1 + (a_2 + \cdots + (a_{n-1} + a_n) + \cdots))$$

respectively. Having in mind $(*)$ for $+$ we put term-substitutions (1) (below).

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## 2. $(+, S)$−terms

Denote by $\Gamma$ a collection of constant symbols, called *initial elements*. Symbol $\emptyset$ is also an element of $\Gamma$. Let $L$ be a language whose elements are all elements of $\Gamma$ and also the operational symbols $+$ and $S$ ($+$ is a binary, $S$ a unary operational symbol). The symbols $x, y, z, u, x', y', z', u', \dots$ are used as variables. We define *terms* by the generalized inductive definition:

(i) Any element of $\Gamma$ is a term.
(ii) A variable is a term.
(iii) If $A, B$ are terms, then the words $(A + B)$, $S(A)$ are terms.

Next we introduce the following term-substitutions

$$
(1)\qquad
\begin{array}{rr}
(i) & (\tau_1 + \tau_1) \to \tau_1 \\
(i') & \tau_1 \to (\tau_1 + \tau_1) \\
(ii) & (\tau_1 + \tau_2) \to (\tau_2 + \tau_1) \\
(iii) & ((\tau_1 + \tau_2) + \tau_3) \to (\tau_1 + (\tau_2 + \tau_3)) \\
(iii') & (\tau_1 + (\tau_2 + \tau_3)) \to ((\tau_1 + \tau_2) + \tau_3) \\
(iv) & \tau_1 \to \tau_1
\end{array}
$$

where $\tau_1, \tau_2, \tau_3$ are any terms.

Let $\sigma_1 \to \sigma_2$ be a substitution. Suppose that $t_1$ is a term which contains, at some place[1] a subterm of the form $\sigma_1$. Replacing $\sigma_1$ with $\sigma_2$ at such a place from $t_1$ we obtain a new term denoted by $t_2$. We shall say that $t_2$ *is neighbouring to* $t_1$. For the sentence: $t_2$ *is neighbouring to* $t_1$ we shall use the following notation: $t_1 \to t_2$. For instance, let $t_1$ be a term

$$S(S(x + S(a + (b + c))) + S(a + (b + c)))$$

and let the substitution $\sigma_1 \to \sigma_2$ be of the form $(\tau_1 + \tau_2) \to (\tau_2 + \tau_1)$. In the given term $t_1$ there are six subterms of the form $(\tau_1 + \tau_2)$, since the symbol $+$ appears in $t_1$ just six times. Let us consider the subterm $(a + (b + c))$, at its first occurrence. Then $\tau_1$ is $a$, and $\tau_2$ is $(b + c)$. By applying the substitution $(\tau_1 + \tau_2) \to (\tau_2 + \tau_1)$ from term $t_1$ we obtain the term $t_2$

$$S(S(x + S((b + c) + a)) + S(a + (b + c)))$$

which is neighbouring to $t_1$. Due to (1)$(iv)$ any term $t$ is neighbouring to $t$, i.e., $t \to t$. Next, we shall define a relation $=$ ('equality') between two terms $t_1, t_2$. Namely:

(2) We shall say that $t_1 = t_2$ holds iff *either* the word $t_1$ is literally equal to the word $t_2$ *or* there are terms $\tau_1, \tau_2, \dots, \tau_k$ such that $\tau_1$ is $t_1$, $\tau_k$ is $t_2$, and each $\tau_i$ $(1 < i \leqslant k)$ is neighbouring to $\tau_{i-1}$

In other words, $t_1 = t_2$ holds iff we have the following 'substitutional chain'

$$(3')\qquad \tau_1 \to \tau_2 \to \tau_3 \to \cdots \to \tau_k \quad (\tau_1 \text{ is } t_1,\ \tau_k \text{ is } t_2)$$

[1]Starting, say, with certain $i$-th letter of the word $t_1$

If the relation $t_1 = t_2$ holds we shall say that $t_1$ *is equal to* $t_2$. Obviously relation $=$ has the following properties

$$\begin{array}{ll} & \tau_1 = \tau_1 \\ & \text{If } \tau_1 = \tau_2, \text{ then } \tau_2 = \tau_1 \\ (3) & \text{If } \tau_1 = \tau_2 \text{ and } \tau_2 = \tau_3, \text{ then } \tau_1 = \tau_3 \\ & \text{If } \tau_1 = \tau_2, \text{ then } S(\tau_1) = S(\tau_2) \\ & \text{If } \tau_1 = \tau_2 \text{ and } \tau_3 = \tau_4, \text{ then } (\tau_1 + \tau_3) = (\tau_2 + \tau_4) \end{array}$$

REMARK 1. Relation $=$ can be defined by equational axioms $x+x=x$, $x+y=y+x$, $(x+y)+z=x+(y+z)$. Namely, it can be easily proved that relation $t_1 = t_2$ holds *iff* formula $t_1 = t_2$ is an equational consequence of the axioms, where $x, y, z$ can be any $(S,+)$-terms.

Now we consider terms of the form

$$A_1, \quad (A_1 + A_2), \quad ((A_1 + A_2) + A_3), \quad (((A_1 + A_2) + A_3) + A_4), \dots$$

For them we shall use the notation

$$A_1, \quad A_1 + A_2, \quad A_1 + A_2 + A_3, \quad A_1 + A_2 + A_3 + A_4, \dots$$

respectively and call them *sum-terms*. Suppose that by $\text{term}(+, A_1, \dots, A_k)$ any term is denoted which is built up from its subterms $A_1, \dots, A_k$ by use of the operational symbol $+$ only. The order of these $A_i$ is not important. Then the following equality holds:

$$(*_1) \qquad \text{term}(+, A_1, \dots, A_k) = A_1 + \cdots + A_k$$

This equality holds in virtue of substitutions (1) of the form (*ii*), (*iii*), (*iii′*). For instance, the equality $((A + (C + B)) + (E + D)) = A + B + C + D + E$ is true.

Let $A_1 + \cdots + A_k$, $B_1 + \cdots + B_q$ be any sum-terms. We define relation $=_S$ between them:

(4) $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$ holds iff each $A_i$ is equal to some $B_j$ and each $B_k$ is equal to some $A_l$

The relation $=_S$ has the following properties

(5) (*i*) $A_1 + \cdots + A_p =_S A_{f(1)} + \cdots + A_{f(p)}$, where $f$ is any permutation of indexes $1, \dots, p$.

(*ii*) $A_1 + \cdots + A_i + \cdots + A_p =_S A_1 + \cdots + A_i + A_i + \cdots + A_p$, where the right hand side is obtained from the left hand side by replacing $A_i$ by $A_i + A_i$

(*ii′*) $A_1 + \cdots + A_i + A_i + \cdots + A_p =_S A_1 + \cdots + A_i + \cdots + A_p$, where the left hand side is obtained from the right hand side by replacing $A_i$ by $A_i + A_i$

(*iii*) If $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$, then $B_1 + \cdots + B_q =_S A_1 + \cdots + A_p$

(*iv*) If $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$ and $B_1 + \cdots + B_q =_S C_1 + \cdots + C_r$, then $A_1 + \cdots + A_p =_S C_1 + \cdots + C_r$

(*v*) $A_1 + \cdots + A_i + \cdots + A_p =_S A_1 + \cdots + B + \cdots + A_p$ assuming that $A_i$ is replaced by $B$ and $A_i = B$

Assertion (5) is an immediate consequence of definition (4). Relation $=_S$ has also this property

$(*_2)$ If $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$, then $A_1 + \cdots + A_p = B_1 + \cdots + B_q$

which is an immediate consequence of (4) and definition of $=$, i.e., of (2). For instance, $A + B + C =_S B + C + A + C$ holds. Also $A + B + C = B + C + A + C$ holds, which can be proved easily by substitutions (1).

Related to implication $(*_2)$ it is important to know when the opposite implication is true too. In connection with it we introduce the notion of *full sum-term*. Namely, we shall say that $A_1 + \cdots + A_k$ is a *full sum-term* if none of $A_i$ is of the form $(P + Q)$ for some $P$, $Q$. Obviously, any sum-term is equal to certain full sum-term.

Now we shall prove

$(*_3)$ If $B_1 + \cdots + B_q$ is a full sum-term, which is neighbouring to a full sum-term $A_1 + \cdots + A_p$, then $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$ holds.

Indeed, denote by $\sigma$ a substitution of type (1) by which from $A_1 + \cdots + A_p$ we obtain $B_1 + \cdots + B_q$. We distinguish two cases:

1° $\sigma$ is related to some subterm of $A_1 \cdots + A_p$ whose $+$ is inside one $A_i$.
2° $\sigma$ is related to one of $p-1$ symbols $+$ occuring in the sum-term $A_1+A_2+\cdots+A_p$

In the first case applying $\sigma$ to $A_i$ we obtain some $B$, such that $A_i = B$. Then in virtue of (5)($v$) we conclude that

$$A_1 + \cdots + A_{i-1} + A_i + \cdots + A_p =_S A_1 + \cdots + A_{i-1} + B + \cdots + A_p$$

and proof is complete in the first case.

In the second case, having in mind (5)($i$), ($ii$), ($ii'$) the proof completes.

Now we shall prove a generalization of $(*_3)$

$(*_4)$ If $A_1 + \cdots + A_p = B_1 + \cdots + B_q$ holds, where $A_1 + \cdots + A_p$, $B_1 + \cdots + B_q$ are full sum-terms, then also $A_1 + \cdots + A_p =_S B_1 + \cdots + B_q$ holds.

Indeed, let $A_1 + \cdots + A_p = B_1 + \cdots + B_q$. Then, like (3′), there is certain substitutional chain

$$\tau_1 \to \tau_2 \to \cdots \to \tau_k \quad (\tau_1 \text{ is } A_1 + \cdots + A_p,\ \tau_k \text{ is } B_1 + \cdots + B_q)$$

In virtue of $(*_3)$ we have $\tau_1 =_S \tau_2$, $\tau_2 =_S \tau_3, \dots, \tau_{k-1} =_S \tau_k$. Having in mind (5)($iv$) we conclude $\tau_1 =_S \tau_k$ and the proof completes.

Lemma 1. *Suppose that $A_1 + \cdots + A_p$, $B_1 + \cdots + B_q$ are full sum-terms. Then the following equivalence is true*

$$A_1 + \cdots + A_p = B_1 + \cdots + B_q \text{ holds iff each } A_i \text{ is equal to some } B_j \text{ and each } B_k \text{ is equal to some } A_l$$

Proof. Proof follows immediately from $(*_3)$ and (4). For instance, if $p = 2$, $q = 2$ then we have the following equivalence

$(*_5)$ $A_1 + A_2 = B_1 + B_2$ iff $(A_1 = B_1 \wedge A_2 = B_2) \vee (A_1 = B_2 \wedge A_2 = B_1)$ □

LEMMA 2. *$S(A) = S(B)$ iff $A = B$.*

Proof follows directly from the definition (1).

Now we define the notion of *monomial*. This is a term $t$ which is not of the form $(P+Q)$ for some $P, Q$. Let $t$ be any term. It is a monomial just in three cases: $t$ is an initial element or $t$ is a variable or $t$ has the form $S(P)$ for some $P$. Term of the form $S(P)$ will be called *$S$-monomial*. An equality of the form $m_1 = m_2$, where $m_1, m_2$ are monomials will be called *a monomial equality*. Such an equality will be called *a reduced monomial equality* if at least one of $m_1, m_2$ is not an $S$-monomial.

LEMMA 3. *Let $m_1 = m_2$ be a reduced monomial equality. This equality holds iff $m_1, m_2$ are equal as words.*

As a matter of fact, Lemma 1, Lemma 2 and Lemma 3 describe an algorithm, called Set-alg, by which one can decide whether any given terms $t_1, t_2$ are equal or not. Namely, to given equality $t_1 = t_2$ we apply Lemma 1 or Lemma 2 as many times as possible. At the end we obtain some logical expression Expr, built from certain reduced monomial equations $m_i = m_j$ using logical connectives $\wedge$ and $\vee$. Having in mind Lemma 3 we can calculate logical value of Expr. If the obtained value is $\top$, then the equality $t_1 = t_2$ holds. If the value is $\bot$, then the equality $t_1 = t_2$ does not hold. We shall illustrate this algorithm by some examples.

EXAMPLE 1. Let $a, b, c, d$ be some initial elements. Calculate the logical value of a given equality:

$$1^\circ\ S(a+b) = S(b+a), \quad 2^\circ\ S(a+b) = S(b+\emptyset)$$
$$3^\circ\ S(a+S(b+c+d)+b) = S(b+a+S(d+c+b))$$

*Solution.* $1^\circ$ We have the following equivalence-chain

$S(a+b) = S(b+a)$
iff $a+b = b+a$ (By Lemma 2)
iff $(a = b \wedge b = a) \ \vee\ (a = a \wedge b = b)$ (By $(*_5)$, i.e. by Lemma 1)

The answer is *yes* since by Lemma 3 the equalities $a = a$ and $b = b$ are true.

$2^\circ$ We have the following equivalence-chain

$S(a+b) = S(b+\emptyset)$
iff $a+b = b+\emptyset$ (By Lemma 2)
iff $(a = b \wedge b = \emptyset) \ \vee\ (a = \emptyset \wedge b = b)$ (By $(*)$, i.e. by Lemma 1)

The answer is *no* since by Lemma 3 equalities $a = b$, $a = \emptyset$ and $b = \emptyset$ are false.

$3^\circ$ By Lemma 2 we see that the given equality reduces to

$$a + S(b+c+d) + b = b + a + S(d+c+b)$$

Applying Lemma 1 this equality reduces to $S(b+c+d) = S(d+c+b)$. Applying Lemma 2 this equality reduces to $b+c+d = d+c+b$. Finally applying Lemma 1 and Lemma 3 we conclude that the answer is *yes*.

Remark 2. Here, in brief, we describe an algorithm, called Calc, which is simpler than Set-alg. Let a term $S_1$ have the form $S(p_1 + p_2 + \cdots + p_m)$, where $p_i$ are some initial elements. We shall say that such a term is *countable*. Term $S_1$ can be equal to certain other countable term, say $S(q_1 + q_2 + \cdots + q_n)$, where $q_j$ are some initial elements. In such case each element $p_i$ must be the same as some $q_j$ and also each $q_k$ must be the same as some $p_r$. For instance, if $a, b, c$ are initial elements we have equality

$$S(a + b + a + b + c) = S(c + b + a + a)$$

Suppose now that a term $t_1$ has certain countable subterm $S_1$, and that a 'list' $S_1, S_2, \dots S_p$ contains all countable subterms of $t$, which are equal to $S_1$. Suppose that $t_1$ equals $t_2$. Then like $(3')$ we have the following substitutional chain

$$(\sigma_1) \qquad \tau_1 \to \tau_2 \to \tau_3 \to \cdots \to \tau_s \qquad (\tau_1 \text{ is } t_1,\ \tau_s \text{ is } t_2)$$

In the first step $\tau_1 \to \tau_2$ of this chain, any $S_i$ remains unchanged or transforms to some countable term $S'_i$, which is equal to $S_i$. The same holds for other steps. Denote by $S_1, \dots S_p, \dots S_P$ all terms occurring in $(\sigma_1)$ which are equal to $S_1$.

Let $C$ be an initial element not occurring in the chain $(\sigma_1)$. In this chain replace all $S_1, \dots S_p, \dots S_P$ by $C$. If $t$ is any term, by $t\langle C\rangle$ we denote the term obtained by that replacement. In such a way from $(\sigma_1)$ we obtain the following 'formal chain'

$$(\sigma_2) \qquad \tau_1\langle C\rangle \to \tau_2\langle C\rangle \to \tau_3\langle C\rangle \to \cdots \to \tau_s\langle C\rangle$$

Obviously $\tau_1\langle C\rangle, \dots \tau_s\langle C\rangle$ are well-formed $(S, +)$-terms and in addition to that $(\sigma_2)$ is a valid substitutional chain. So we conclude the following

$$\text{If } t_1 = t_2, \text{ then } t_1\langle C\rangle = t_2\langle C\rangle.$$

Now suppose that in $(\sigma_2)$ every $C$ is replaced by any terms $S_1, \dots S_p, \dots S_P$. Then from $(\sigma_2)$ we shall obtain a chain which can be easily extended to a valid substitutional chain[2]. So, we conclude also

$$\text{If } t_1\langle C\rangle = t_2\langle C\rangle, \text{ then } t_1 = t_2.$$

The mentioned algorithm *Calc* is based on the equivalence

$$(\text{Cal}) \qquad t_1 = t_2 \quad \text{iff} \quad t_1\langle C\rangle = t_2\langle C\rangle$$

We illustrate Calc by two examples. First one is: prove or disprove the equality

$$S(a + S(S(a) + b) + S(b + S(a)) + c) = S(a + c + S(b + S(a)))$$

Using (Cal) we have the following equivalence-chain

$S(a + S(S(a) + b) + S(b + S(a)) + c) = S(a + c + S(b + S(a)))$

iff $S(a + S(p + b) + S(b + p) + c) = S(a + c + S(b + p))$ $\quad$ ($S(a)$ is replaced by $p$)

iff $S(a + q + q + c) = S(a + c + q)$ $\quad$ ($S(p + b)$, $S(b + p)$ are replaced by $q$)

iff $r = r$ $\quad$ ($S(a + q + q + c)$, $S(a + c + q)$ are replaced by $r$)

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[2]For instance, if in the chain $S(a + C) \to S(C + a)$ we replace the first $C$ by $S(p + q)$ and the second by $S(q + p)$ we obtain $S(a + S(p + q)) \to S(S(q + p) + a)$, which can be extended to this valid substitutional chain $S(a + S(p + q)) \to S(S(p + q) + a) \to S(S(q + p) + a)$.

So, the given equality is proven. Here $p, q, r$ are initial elements.

The second example is: prove or disprove the equality

$$S(a + S(S(b)) + c) = S(c + a + S(S(b) + d))$$

Using (Cal) we have the following equivalence-chain

$S(a + S(S(b)) + c) = S(c + a + S(S(b) + d))$

iff $S(a + S(p) + c) = S(c + a + S(p + d))$ ($S(b)$ is replaced by $p$)

iff $S(a + q + c) = S(c + a + S(p + d))$ ($S(p)$ is replaced by $q$.)

As a matter of fact, because $S(p)$ does not appear on the right hand side we conclude that the last equality is false. Consequently the given equality is false.

Notice that the name Calc is related to the fact that this algorithm in some sense 'calculates' the given terms. Terms of the form $S(p_1 + p_2 + \cdots + p_m)$, where $p_i$ are some initial elements, are called *countable*, since the algorithm *Calc* is 'able to calculate just them'.

### 3. Definition of finite sets

A *ground* term is a term not containing variables. Now we define the notion of a finite set:

(6) A finite set is either $\emptyset$ (called the empty set) or a ground term of the form $S(A)$ (called a non-empty set).

According to Lemma 1 any set $A$ can be expressed in one of the forms

$(*_6)$ $$1^\circ:\ \emptyset, \qquad 2^\circ:\ S(A_1 + \cdots + A_n)$$

where $n = 1, 2, \ldots$ and $A_1 + \cdots + A_n$ is a full sum-term. One may suppose that $A_i$ are pairwise different terms. These forms are called *the canonical forms* for set $A$. By virtue of Lemma 1 the form $(*_6)$ $2^\circ$ is unique up to the order of $A_1, A_2, \ldots, A_n$. In the sequel we always assume that finite sets are given in a canonical form.

EXAMPLE 2. Let $1, 2, 3$ be initial elements. Then the ground term $S((1+3)+(2+3))$ is a finite set. One of its canonical forms is $S(1+2+3)$. Besides this there are also 5 others canonical forms

$$S(1+3+2),\ S(2+1+3),\ S(2+3+1),\ S(3+1+2),\ S(3+2+1)$$

which differ only in the order of $1, 2, 3$.

Concerning the given definition (6) of a finite set we point out that a finite set is defined as *a well defined term*. Consequently, we can produce various recursive definitions for them.

First, we define the relation $\in$:

(7) $x \in S(A)$ iff $x = A$

$x \in S(A_1 + A_2 + \cdots + A_n)$ iff $x = A_1$ or $x \in S(A_2 + \cdots + A_n)$ $(n > 1)$

$x \notin \emptyset$

For instance, $2 \in S(1+2+3)$. Indeed:

$$\begin{aligned} 2 \in S(1+2+3) \ &\text{iff} \ \ 2 = 1 \text{ or } 2 \in S(2+3) \\ &\text{iff} \ \ 2 \in S(2+3) \qquad \text{(Since } 2 = 1 \text{ is false)} \\ &\text{iff} \ \ 2 = 2 \text{ or } 2 \in S(3) \\ &\text{iff} \ \ 2 = 2 \qquad \text{(Since } 2 = 2 \text{ is true)} \end{aligned}$$

So, it is true that $2 \in S(1+2+3)$. But, for instance $2 \notin S(1+3)$. Indeed:

$$\begin{aligned} 2 \in S(1+3) \ &\text{iff} \ \ 2 = 1 \text{ or } 2 \in S(3) \\ &\text{iff} \ \ 2 \in S(3) \qquad \text{(Since } 2 = 1 \text{ is false)} \\ &\text{iff} \ \ 2 = 3 \end{aligned}$$

Since the equality $2 = 3$ is false we conclude that $2 \in S(1+3)$ is false too.

Notice that in general if $A_1, \dots, A_n$ are some given ground terms then $S(A_1 + \dots + A_n)$ is the set whose all elements are $A_1, \dots, A_n$.

Bearing in mind Lemma 1 and the definition (7) one can easily prove the following well known equivalence (*Extensionality axiom in ZF set theory*)

$$A = B \ \leftrightarrow \ (\forall x)(x \in A \leftrightarrow x \in B)$$

where $A, B$ are any finite sets and the variable $x$ ranges over ground terms only.

The next step is to define $|A|$–the cardinal number of the set $A$. By use of the 'ordinary' notion of natural number we have the inductive definition

$$|\emptyset| = 0, \quad |S(A)| = 1, \quad |S(A_1 + A_2 + \dots + A_n)| = 1 + |S(A_2 + \dots + A_n)|.$$

Now we give several definitions, and each of them will be syntactical; in other words for each of them we can make a corresponding *decision algorithm*.

(Relation $\subseteq$) $A \subseteq B$ iff $(\forall x)(x \in A \Rightarrow x \in B)$
(Operation $\cup$) $\emptyset \cup x = x$ ($x$ is any finite set)
$S(A_1) \cup S(B_1 + \dots + B_m)$ is $S(A_1 + B_1 + \dots + B_m)$
$S(A_1 + \dots + A_n) \cup S(B_1 + \dots + B_m)$ is $S(A_2 + \dots + A_n) \cup S(B_1 + \dots + B_m)$ if $A_1 \in S(B_1 + \dots + B_m)$,
otherwise it is $S(A_1) \cup (S(A_2 + \dots + A_n) \cup S(B_1 + \dots + B_m))$
(Operation $\cap$) $\emptyset \cap x = \emptyset$ ($x$ is any finite set)
$S(A_1 + \dots + A_n) \cap S(B_1 + \dots + B_m)$ is $S(A_2 + \dots + A_n) \cap S(B_1 + \dots + B_m)$ if $A_1 \notin S(B_1 + \dots + B_m)$,
otherwise it is $S(A_1) \cup \big(S(A_2 + \dots + A_n) \cap S(B_1 + \dots + B_m)\big)$
(Operation $\smallsetminus$) $\emptyset \smallsetminus x = \emptyset$ ($x$ is any finite set)
$S(A_1 + \dots + A_n) \smallsetminus S(B_1 + \dots + B_m)$ is $S(A_2 + \dots + A_n) \smallsetminus S(B_1 + \dots + B_m)$ if $A_1 \in S(B_1 + \dots + B_m)$,
otherwise it is $S(A_1) \cup \big(S(A_2 + \dots + A_n) \smallsetminus S(B_1 + \dots + B_m)\big)$

Now we shall define the ordered pair of two member terms $x, y$, in symbols $\Pi(x, y)$. There are two ways to do this: one is to extend $\Omega$ by a new binary operational symbol $\Pi$ and to add no new axiom concerning $\Pi$, and to regard term

$\Pi(x,y)$ as the ordered pair of $x$ and $y$. The alternative way is to adopt Wiener–Kuratowski idea, i.e., to introduce the following definition

$$\Pi(x,y) = S(S(x) + S(x+y))$$

Further, we can in the usual way define cartesian product $X \times Y$ of two sets, of more sets, a binary, ternary,... relation, a function $f : X \to Y$, etc. Again and again each of such notions is determined by some ground term.

EXAMPLE 3. Let $A = S(1+2+3)$, $B = S(a+b)$. Then

$$A \times B = S(\Pi(1,a) + \Pi(1,b) + \Pi(2,a) + \Pi(2,b) + \Pi(3,a) + \Pi(3,b))$$

The term $S(\Pi(1,a) + \Pi(2,b) + \Pi(3,a))$ determines a function $f : A \to B$ such that $f(1) = a$, $f(2) = b$, $f(3) = a$.

We have already mentioned that the finite sets defined by (6) satisfy Extensionality axiom of ZF set theory. It is not difficult to see that, in Fin-set, except *Axiom of infinity*, all other axioms of ZF theory can be proved in a simple way. Moreover, if some axioms say that there exist some sets $x, y, \dots$, then one can make 'algorithmic proof', which effectively construct such sets $x, y, \dots$

We illustrate this idea by considering *Subset axioms*. So, let $t = S(a_1 + \cdots + a_n)$ be a given set and $\phi(x)$ any given formula, condition containing $x$ as a free variable. Suppose that for each $a_i$ we can determine whether $\phi(a_i)$ is true or false. We should prove that there exists a set $T$ such that: $x \in T$ iff $x \in t \wedge \phi(x)$.

Denote by $b_1, \dots, b_k$ all of those $a_i$ for which $\phi(a_i)$ is true. If $k = 0$, then $T = \emptyset$, otherwise $T = S(b_1 + \cdots + b_k)$. The proof is complete.

EXAMPLE 4. Let $t = S(1+2+3+4+5)$ and let $\phi(x)$ be the following condition $x \in S(1+4+8+9)$. Then $T = S(1+4)$. However, suppose that we extend the language $\Omega$ by a relation symbol ev ('to be even') and also add the following new axioms ev(2), ev(4), ev(6). Let $\phi(x)$ be the formula ev($x$). Then the corresponding $T$ is $S(2+4)$.

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