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A NEW SOLVING PROCEDURE BY m-M CALCULUS FOR PROBLEMS OF CONSTRAINED OPTIMIZATION*

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Abstract: In this paper we state two procedures for constrained optimization based on the concepts of m-M Calculus. The first procedure is called *basic* and the second is called *quick* solving procedure. The quick solving procedure is very effective. It can also be applied to problems of unconstrained optimization.

Keywords: Constrained optimization, procedure, m-M Calculus.

1. INTRODUCTION

At the beginning we briefly state some basic facts of the m-M Calculus. The m-M Calculus deals with the so-called m-M functions, i.e. functions of the form

$$f : D \to \mathbb{R} \quad (D = [a_1, b_1] \times \cdots \times [a_n, b_n], \text{ where } n > 0 \text{ is any element of } \mathbb{N}; \text{ and } a_i, b_i \in \mathbb{R})$$

subject to the following supposition:

For each n-dimensional segment $\Delta = [\alpha_1, \beta_1] \times \cdots \times [\alpha_n, \beta_n] \subset D$ *a pair of real numbers*, denoted by $m(f)(\Delta), M(f)(\Delta)$, satisfying the conditions

$$m(f)(\Delta) \le f(X) \le M(f)(\Delta) \quad (\text{for all } \Delta \subset D,\ X \in \Delta) \tag{1}$$

$$\lim_{diam\Delta \to 0}(M(f)(\Delta) - m(f)(\Delta)) = 0 \quad (\text{where } diam\Delta := (\sum(\beta_i - \alpha_i)^2)^{1/2}) \tag{2}$$

is effectively given.

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Such an ordered pair $\langle m(f), M(f)\rangle$ of mappings $m(f), M(f)$ (both mapping the set of all $\Delta \subset D$ into $\mathbb{R}$) is called an $m - M$ pair for the function *f*. We also say that $m(f), M(f)$ are *generalized minimum* and *maximum* for *f* respectively. For instance, with only a few exceptions all elementary functions are $m - M$ functions (see Lemma 1.4 in [2]). The conditions (1) and (2) are taken as axioms of the m-M Calculus.

From (1) and (2) one can easily prove the following *lim*-equations

$$\lim M(f)(\Delta) = \lim m(f)(\Delta) = f(X_0)$$
$$\text{(where } diam\Delta \to 0 \text{ and } X_0 \text{ is a joint point for all these } \Delta\text{)} \qquad (3)$$

Now we consider one problem of constrained optimization.

Problem 1. *Let* $f, g : [a,b] \to \mathbb{R}$ *be given* $m - M$ *functions. Denote by A the set of all* $x \in [a,b]$ *satisfying the condition*

$$f(x) \geq 0$$

Restricting the function g to the set A we seek the set B of all $x \in A$ *at which g attains minimum value.*

In the solving procedure we shall use the notion of[1] *cell-decomposition* of some *n*-dimensional segment[2] $D \subset \mathbb{R}^n$. In case of Problem 1 we have $n = 1$ and *D* is $[a,b]$. One cell-decomposition is called *dyadic*. Then, in the first step we have the segment $[a,b]$ while in the second step we have two 'cells' $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ etc.

Generally, by $\Delta_i(x)$ we denote the cell, which belongs to the set of all cells in the i^{th} step and also contains *x*. By using logical formulas Problem 1 can be stated in this way.

Find all $x \in [a,b]$ such that the formula

$(\forall y \in [a,b])(f(y) \geq 0 \Rightarrow (g(y) \geq g(x),\ f(x) \geq 0))$ holds.

Using the well known tautology $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ we can eliminate[3] the implication symbol $\Rightarrow$ from that logical formula. However this idea is not practical enough, since the numbers of feasible cells, related to the solutions *x*, can be rather big. The main reason is: by definition[4] the implication $p \Rightarrow q$ is the true if p is **false**.

In order to avoid this shortcoming of the (material) implication we shall use the following idea:

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[1] We use the word cell in the sence: *subsegment*.

[2] It is similar to that in the definition of Riemann integrals. See Definition 1.2 in [2].

[3] Such a way is used in [2], p. 69.

[4] Notice, that this definition is for the so called the *material implication*.

Step-by-step, as it is customary in the m-M Calculus, we approximately build the set *A*. Therefore, at the same time, based on this building in each step we separate certain cells by which we approximately build the set *B*.

In spirit of this we have the following consideration:

Any $y \in A$ should have property $f(y) \geq 0$, whence it follows the $m-M$ condition

$M(f)(\Delta) \geq 0$

where Δ is any cell containing y.

Next, any $x \in B$ should satisfy the conditions

$g(y) \geq g(x)$ for any $y \in A$, and $f(x) \geq 0$

From the first one we conclude the following $m-M$ condition

$$M(g)(\Delta) \geq m(g)(\Delta(x)) \quad \text{for any } \Delta \subseteq A \tag{*1}$$

where $x \in \Delta(x)$. The second condition $f(x) \geq 0$ means that we should make certain that *x* must be an element of the set *A*.

Now, we are going to describe the so called *basic* solving procedure for Problem 1. There are two cases: *A* is the empty set, or *A* is not the empty set. In the first case the solving procedure is finite, with the answer: Problem 1 is impossible.

In the second case the solving procedure is infinite. Then we can accomplish only a finite number of steps of the procedure and consequently we obtain an approximate solution. To achieve this we use two set-sequences: $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$, which are unions of some segments. Using the idea of induction, we define their members as follows:

The initial members are defined by $A_1 = D,\ B_1 = D$, where D is $[a,b]$. If $M(f)(D) < 0$ then the basic procedure halts with the answer: Problem 1 is impossible.

Suppose that we have defined A_i, B_i for some *i*. Then we define A_{i+1}, B_{i+1} in the following manner:
First, we decompose A_i into some cells, say

$\alpha_1, \alpha_2, \ldots, \alpha_k$.

Next, denote by

$\alpha_1', \alpha_2', \ldots, \alpha_r'$.

all members α_i $(i = 1, \ldots, k)$ which satisfy the condition $M(f)(\alpha_i) \geq 0$. We shall say that they are *f*-feasible cells. Then, A_{i+1} is defined as the union of all such *f*-feasible cells. If A_{i+1} is the empty set, then the basic procedure halts with the answer: Problem 1 is impossible.

Let α' be any element of α'_i where $i = 1,...,r$. We shall say that α' is g-feasible if it satisfies (see (*1)) the following condition:

$$mg(\alpha') \leq M(g)(\alpha'_j) \quad \text{for all } j = 1,...,r . \qquad (*2)$$

Then, B_{i+1} is defined as the union of all g-feasible sub segments.

To complete the description of the basic solving procedure we prove the following equalities

$$A = \bigcap_{i \in N} A_i, \quad B = \bigcap_{i \in N} B_i .$$

The first equality is one example of equalities which appeared in Theorem 2.1 (in [2]).

We now pass to the second equality. We shall prove that every element *x* of the set *B* is an element of the intersection set $\bigcap B_i$. Consider the sequence $\Delta_i(x)$ $(i = 1,2,...)$. Since, $x \in B$ we have $x \in A$. Consequently $\Delta_i(x)$ is *f*-feasible. Next, thanks to the minimum property of *x* we easily conclude that $\Delta_i(x)$ is also g-feasible. Thus, $\Delta_i(x)$ is a part, a subset of each B_i. Consequently $\bigcap B_i$ must contain *x*. In this way we have proved the relation $B \subseteq \bigcap B_i$.

To complete the proof we should prove that every element *z* of $\bigcap B_i$ must be an element of the set *B*. Suppose the contrary. Then there is an element $x_0 \in A$ such that the inequality

$$g(z) > g(x_0) \qquad (\phi)$$

holds. Consider the sequence of cells $\Delta_i(x_0)$, $\Delta_i(z)$, the sub segments from the i^{th} step which contain x_0 and z, where the first one is a subset of A_i, and the second one a subset of B_i. Having in mind (*2) we conclude that for every *i* holds:

$$m(g)(\Delta_i(z)) \leq M(g)(\Delta_i(x_0)) .$$

Having in mind (3), from (ϕ) we obtain that for some *i* the inequality

$$m(g)(\Delta_i(z)) > M(g)(\Delta_i(x_0))$$

holds, which contradicts to the previous inequality. In such a way we have completed the proof and also ended the description of the basic solving procedure (for Problem 1).

In the basic solving procedure we have not used the following fact concerning the definition of the minimum value of any function, as the function g appeared in Problem 1:

Assume that the set *A* is non empty. Then, if *P* is any over-set of the set *B* the following *min*-equation

$$\min_g(P) = B$$

holds, where $\min_g(P)$ is the set of all points from *P* at which the function g, (i.e. its corresponding restriction), attains the minimum value.

Using this idea now we shall describe the second solving procedure, called the *quick solving procedure*. We use only set-sequence $B_1, B_2, \ldots,$ which is a union of some cells, sub segments. Using the idea of induction we define its members as follows:

The initial member B_1 is defined by $B_1 = D$, where D is $[a,b]$. If $M(f)(D) < 0$ than the quick procedure halts with the answer: Problem 1 is impossible.

Suppose that we have defined B_i for some *i*. Then we define B_{i+1} in the following manner:

First, we decompose B_i into some cells, say

$\alpha_1, \alpha_2, \ldots, \alpha_k$

Next, denote by

$\alpha'_1, \alpha'_2, \ldots, \alpha'_r$.

all α_i $(i = 1, \ldots, k)$ which satisfy the condition $M(f)(\alpha_i) \geq 0$. If there is no such cell then the quick procedure halts with the answer: Problem 1 is impossible.

Let α' be any of α'_j where $j = 1, \ldots, r$. We say that α' is *g*-feasible if it satisfies the condition:

$mg(\alpha') \leq M(g)(\alpha'_j)$ for all $j = 1, \ldots, r$.

Then, B_{i+1} is defined as the union of all such *g*-feasible cells.

To complete the description of the quick solving procedure we prove the equation

$$B = \bigcap_{i \in N} B_i .$$

Let *x* be any element of *B*. Consider the sequence of cells $\Delta_i(x)$ containing *x*. Thanks to the minimum property of *x*, this $\Delta_i(x)$ in every step *i* belongs to the *g*-feasible cells. Consequently the intersection set of B_i must contain such *x*. So, we have proved that *B* is a subset of the intersection set for sequence $B_1, B_2, \ldots$

Now we prove that set $\bigcap B_i$ contains only elements of *B*. Suppose the contrary that there is certain *y* so that $y \notin B$ and $y \in \bigcap B_i$. Denote by *x* an element of *B*. This *y* satisfies the inequality $g(y) > g(x)$. Denote by *d* the difference $g(y) - g(x)$. Now we consider the cell-sequences $\Delta_i(x)$, $\Delta_i(y)$, where $i = 1, 2, \ldots$ By virtue of (3) there is $i_0 \in \mathbb{N}$ such that the inequalities

$$|M(g)(\Delta_i(x)) - g(x)| < d/2, \quad |m(g)(\Delta_i(y)) - g(y)| < d/2$$

hold for every $i \geq i_0$. Whence we conclude the following inequality

$$m(g)(\Delta_i(y)) > M(g)(\Delta_i(x)) \tag{*3}$$

where $i \geq i_0$ can be arbitrary. According to the definition of the sequence B_i, in every step *i* the cells $\Delta_i(x)$ and $\Delta_i(y)$ must be *g*-feasible. Imagine the moment at which in the

i^{th} step we separate g-feasible cells in order to make B_{i+1}. Then according to (*3) the cell $\Delta_i(y)$ cannot be g-feasible. Since we obtain a contradiction, the proof completes.

Now we pass to the following general problem.

Problem 2. *Let $D \subset \mathbb{R}^n$ be a given n-dimensional segment and $f_{ij}, g : D \to \mathbb{R}$ be given m – M functions, where $i = 1,...,m;\ j = 1,..., p_i$. Denote by A the set of all $x \in D$ satisfying the following condition, i.e. the $(\vee,\wedge)$-formula*

(Constraint)
$$\begin{array}{l} f_{11}(X) \ge 0 \wedge \cdots \wedge f_{1p_1}(X) \ge 0 \\ \vdots \\ f_{m1}(X) \ge 0 \wedge \cdots \wedge f_{mp_m}(X) \ge 0 \end{array}$$

Restricting the function g to the set A we seek the set B of all $X \in A$ at which g attains minimum value.

Some particular cases of Problem 2 are:

Case $m = 1,\ p_1 = 1$. Then (Constraint) reads $f_{11} \ge 0$. This is a case of Problem 1.

Case $m = 1$. Then (Constraint) reads

$$f_{11}(X) \ge 0 \wedge \cdots \wedge f_{1p_1}(X) \ge 0$$

Case $p_1 = p_2 = \cdots = p_m = 1$. Then (Constraint) reads

$$f_{11}(X) \ge 0 \vee \cdots \vee f_{m1}(X) \ge 0$$

Now we shall describe two solving procedures for Problem 2. In fact, these procedures are very close to the basic and the quick procedure for Problem 1.

It is a supposed that some cell-decomposition of the segment *D* is used. Similarly as in Problem 1 by $\Delta_i(X)$ will be denoted that the cell[5] which belongs to the set of all cells in the i^{th} step also contains *X*.

The key role have notions of *F*-feasible and *g*-feasible cells. Definition of *F*-feasible cell Δ reads:

A cell Δ is *F*-feasible if it satisfies the following condition

$$\begin{array}{l} M(f_{11})(\Delta) \ge 0 \wedge \cdots \wedge M(f_{1p_1})(\Delta) \ge 0 \\ \vdots \\ M(f_{m1})(\Delta) \ge 0 \wedge \cdots \wedge M(f_{mp_m})(\Delta) \ge 0 \end{array}$$

Strictly said, having in mind Definition 4.1 in [2], we see that the definiens of the definition of *F*-feasible cells are (a little freely written) just $m(Constraint)(\Delta)$. On the other hand, one can without use of Definition 4.1 directly come to that definiens.

Definition of *g*-feasible cells is almost the same as that in Problem 1.

Next, by use of the notions *F*- and *g*- feasible cells we can define the sequences A_i, B_i, quit similarly as we did in Problem 1.

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[5] i.e. *n*-dimensional subsegment of *D*.

Consequently we can define the basic and the quick procedure for Problem 2. Almost copying the proofs concerning the basic and the quick procedure for Problem 1 we can obtain the proofs for the basic and the quick solving procedure for general Problem 2.

We point out that the quick procedure is very effective and that it can also be applied to problems of unconstrained optimization.

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