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A METHOD FOR SOLVING A CLASS OF CYCLIC FUNCTIONAL EQUATIONS

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1. There exist different results on cyclic functional equations ([1], [2], [3], [4], [5], [6], [7], [8], [9], [10]). We give a method for solving a very general class of those equations.

2. Let S_1, S_2, S_3, be nonempty sets and $L, D: S_2^n \to S_3$ given functions (n-fixed natural number). Consider the following cyclic functional equation:

$$(1)\qquad L(f(x_1, x_2, \ldots, x_n), f(x_2, x_3, \ldots, x_1), \ldots, f(x_n, x_1, \ldots, x_{n-1})) = D(f(x_1, x_2, \ldots, x_n), f(x_2, x_3, \ldots, x_1), \ldots, f(x_n, x_1, \ldots, x_{n-1})),$$

where $f: S_1^n \to S_2$ is unknown function.

We denote this equation as follows:

$$E(f(x_1, x_2, \ldots, x_n), f(x_2, x_3, \ldots, x_1), \ldots, f(x_n, x_1, \ldots, x_{n-1})) \quad (x_1, x_2, \ldots, x_n \in S_1; f(x_1, x_2, \ldots, x_n) \in S_2).$$

Obviously, *the equation* (1) *is possible if and only if the following condition holds:*

$$(2)\qquad (\exists u)\,(u \in S_2 \wedge E(u, u, \ldots, u)),$$

Let S denote a subset of the set S_2^n such that:

$$(u_1, u_2, \ldots u_n) \in S \overset{def}{\Leftrightarrow} (E(u_1, u_2, \ldots, u_n) \wedge E(u_2, u_3, \ldots, u_1) \wedge \ldots \wedge E(u_n, u_1, \ldots, u_{n-1})).$$

If the equation (1) is possible then S is a nonempty set, because:

$$E(u, u, \ldots, u) \Rightarrow (u, u, \ldots, u) \in S.$$

Next, we prove the following fundamental lemma.

Lemma. *If condition* (2) *holds then there exists a function* $F: S_2^n \to S_2$ *such that:*

1° $E(F(u_1, u_2, \ldots, u_n), F(u_2, u_3, \ldots, u_1), \ldots, F(u_n, u_1, \ldots, u_{n-1}))$ (*for all* $u_i \in S_2$);

2° $(u_1, u_2, \ldots, u_n) \in S \Rightarrow F(u_1, u_2, \ldots, u_n) = u_1$,

Proof. Let u_0 be an element of S such that $E(u_0, u_0, \ldots, u_0)$. The function F: $(u_1, u_2, \ldots, u_n) \in S \Rightarrow F(u_1, u_2, \ldots, u_n) \overset{def}{=} u_1$; $(u_1, u_2, \ldots, u_n) \notin S \Rightarrow F(u_1, u_2, \ldots, u_n) \overset{def}{=} u_0$ satisfies the conditions 1° and 2°.

By the lemma we prove the following theorem which solves the problem of determining the general solution of the equation (1).

Theorem. *Let* $F: S_2^n \to S_2$ *be a function satisfying the conditions* 1° *and* 2°. *Then by the formula*

$$(3)\ f(x_1, x_2, \ldots, x_n) = F(\Pi(x_1, x_2, \ldots, x_n), \Pi(x_2, x_3, \ldots, x_1), \ldots, \Pi(x_n, x_1, \ldots, x_{n-1}))$$

$$(\Pi: S_1^n \to S_2 \text{ an arbitrary function})$$

is determined the general solution of the equation (1).

Proof. If $\Pi: S_1^n \to S_2$ then the function f defined by (3) satisfies (1). (This follows from the condition 1°).

Taking f (f is the solution of (1)) instead of Π in (3) we conclude (by 2°):

$$F(f(x_1, x_2, \ldots, x_n), f(x_2, x_3, \ldots, x_1), \ldots, f(x_n, x_1, \ldots, x_{n-1})) = f(x_1, x_2, \ldots, x_n).$$

Accordingly, (3) determines the general solution of the equation (1).

3. We give some examples.

$$\text{(I)}\quad f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_1) + \cdots + f(x_n, x_1, \ldots, x_{n-1}) = 0$$

$$(x_1, x_2, \ldots, x_n; f(x_1, x_2, \ldots, x_n) \text{ real numbers}).$$

In this case, one function F is:

$$F(u_1, u_2, \ldots, u_n) \stackrel{\text{def}}{=} \frac{-(u_1 + u_2 + \cdots + u_n) + n\, u_1}{n}$$

and the general solution of (I) is

$$f(x_1, x_2, \ldots x_n) = \Pi(x_1, x_2, \ldots, x_n) - \frac{1}{n}(\Pi(x_1, x_2, \ldots, x_n) +$$

$$\Pi(x_2, x_3, \ldots, x_1) + \cdots + \Pi(x_n, x_1, \ldots, x_{n-1}))\ (\Pi\text{ -an arbitrary real function}).$$

Similarly, in the case of the general linear cyclic functional equation, there exists the *linear function* F [9].

The equation, [7]:

$$\text{(II)}\ A f^2(x, y) + B f(x, y) f(y, x) + C f^2(y, x) + D f(x, y) + E f(y, x) + F = 0,$$

$$(A, B, C, D, E, F; x, y; f, (x, y) \text{ real numbers}).$$

Let S be the set of all (u, v) such that:

$$A u^2 + Buv + Cv^2 + Du + Ev + F = 0$$

$$A v^2 + Buv + Cu^2 + Dv + Eu + F = 0 \quad (u, v \text{ real numbers}).$$

Determine, for example, the function F in the following way:

$$(u, v) \in S \Rightarrow F(u, v) = u; \quad (u, v) \notin S \Rightarrow F(u, v) = u_0$$

where u_0 is a real number such that:

$$(A + B + C) u_0^2 + (D + E) u_0 + F = 0.$$

Then

$$f(x, y) = F(\Pi(x, y), \Pi(y, x)) \quad (\Pi - \text{an arbitrary function})$$

is the general solution of the equation (II).

The equation (II) is possible if and only if

$$(\exists u)\,(u \text{ a real number} \wedge (A+B+C)\,u^2+(D+E)\,u+F=0).$$

The equation:

(III) $$f^3(x, y)-3f(y, x)+2=0 \quad (x, y, f(x, y) \text{ real numbers}).$$

In this case only (1,1) and (−2, −2) are elements of the set S. One function F is

$$F(-2, -2) \overset{\text{def}}{=} -2; \quad F(u, v) \overset{\text{def}}{=} 1, \quad \text{otherwise.}$$

The general solution of (III) is:

$$f(x, y)=F(\Pi(x, y), \Pi(y, x)) \quad (\Pi \text{—an arbitrary function}).$$

Remark. *The set S_3 may be $\{t, f\}$ (t—„true", f—„false"). Then $E(u_1, u_2, \ldots, u_n)$ is a relation of the set S_2.*

Example. Determine all real functions $f(x, y)$ which satisfy the condition

(IV) $$f(x, y)+f(y, x) \leqslant 0.$$

Solution. The function $F(u, v)$ is determined, for example, by

$$u+v \leqslant 0 \Rightarrow F(u, v) \overset{\text{def}}{=} u; \quad u+v>0 \Rightarrow F(u, v) \overset{\text{def}}{=} 0.$$

$$\text{(For instance, } F(u, v)=u\left(1-\frac{1}{2}(\operatorname{sgn}(u+v)+\operatorname{sgn}|u+v|)\right)\text{).}$$

Then the formula

$$f(x, y)=F(\Pi(x, y), \Pi(y, x)) \quad (\Pi \text{—an arbitrary real function}),$$

gives all functions $f(x, y)$.

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