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A SURVEY OF THE m-M CALCULUS

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To the memory of Professor Jovan Petrić

Abstract: This paper is a brief version of the monography [1]. The m-M Calculus deals with the so-called m-M functions, i.e. functions of the form $f : D \to R$ ($D = [a_1, b_1] \times \ldots \times [a_n, b_n]$, where $n > 0$ is any integer and $a_i, b_i \in R$) subjected to the following supposition:

For each n-dimensional segment $\Delta = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n] \subset D$ a pair of real numbers, denoted by $m(f)(\Delta)$, $M(f)(\Delta)$, satisfying the conditions

$$m(f)(\Delta) \le f(X) \le M(f)(\Delta) \qquad \text{(for all } \Delta \subset D,\ X \in \Delta) \tag{0.1}$$

$$\lim_{diam\,\Delta \to 0} (M(f)(\Delta) - m(f)(\Delta)) = 0 \qquad \text{(where } diam\,\Delta : (\textstyle\sum(\beta_i - \alpha_i)^2)^{1/2}) \tag{0.2}$$

is effectively given.

Such an ordered pair $\langle m(f), M(f) \rangle$ of mappings $m(f), M(f)$ (both mapping the set of all $\Delta \subset D$ into R) is called an **m-M pair** of the function f. We also say that $m(f), M(f)$ are **generalized minimum** and **maximum** for f respectively. For instance, with only few exceptions all elementary functions are m-M functions (Lemma 1.2).

The conditions (0.1) and (0.2) are taken as axioms of the **m-M calculus**. A logical analysis of these axioms is given here and, in addition to the other results, a series of equivalences is proved which enable us to express some relationships for m-M functions by means of the corresponding relationships fot their m-M pairs (see (2.2), (2.5), (2.6), (2.7)). There are many various applications of the m-M calculus, such as

- *Solving systems of inequalities, systems of equations (Section 1)*
- *Finding n-dimensional integrals (Section 1, Example 1.5)*
- *Solving any problem expressed by a positive $\le$ formula (Section 2), among others*
  - *Problem of constrained optimization (Problem 2.2, Problem 2.3)*
  - *Problem of unconstrained optimization (Problem 2.1)*
  - *Problems from Interval Mathematics (Problem 2.4)*

- *Finding functions satisfying a given m-M condition (e.g. functional condition, or difference equation, or differential equation. Section 3).*

As it is well known, by the usual methods of numerical analysis, assuming certain convergence conditions, we aproximately determine, step-by-step, **one** solution of the given problem. However, applying the methods of m-M calculus we aproximately determine **all** solutions of the given problem, and we assume almost nothing about the convergence conditions. The solutions are, as a rule, sought in a prescribed n-dimensional segment D. If the given problem, e.g. a system of equations, has no solutions in D, then applying the method of m-M calculus this can be established at a certain finite step k. The basic methodological idea of the m-M calculus is:

It gives a sufficient condition $Cond(\Delta)$ which ensures that an n-dimensional segment Δ does not contain any solution of the considered problem P. Applying repeatedly this criterion, we reject from the original n-segment D those "pieces" **which do not contain solutions**, so that in the limit case the remaining "pieces" form the set S of all solutions of the problem P (if indeed there is a solution of P).

Keywords: The ideas of m-M calculus are related to some techniques used in global optimization [2,3,4] and interval mathematics [5], but the theory of m-M calculus has much wider range of application.

1. HOW TO FIND AN m-M PAIR OF A GIVEN FUNCTION. APPLICATIONS

Let $f : D \to R$, with $D = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset R^n$, be a given m-M function. As the first fact notice that from axioms (0.1), (0.2) follows that the function f must be continuous. It is easy to see that in some sense the opposite assertion is also true. Namely, if $f : D \to R$ is a given continuous function then one of its m-M pairs may be defined by

$$m(f)(\Delta) = \min_{X \in \Delta} f(X), \quad M(f)(\Delta) = \max_{X \in \Delta} f(X).$$

We point out that this formula can be effectively used in case of functions, monotone in each of its arguments. For instance, we have the following assertion

Proposition 1.1. If $f : [a_1, b_1] \to R$ is a continuous monotone function then one of its m-M pairs is determined by the equality

(i) $m(f)(\Delta) = f(\alpha_1)$, $M(f)(\Delta) = f(\beta_1)$ *if f is nondecreasing* *or*

(ii) $m(f)(\Delta) = f(\beta_1)$, $M(f)(\Delta) = f(\alpha_1)$ *if f is nonincreasing.*

Next we list the following table in which f denotes the function defined by the given expression and $\langle m(f)(\Delta), M(f)(\Delta) \rangle$ is and m-M pair of f.

Table 1.1

Function f	$m(f)(\Delta)$	$M(f)(\Delta)$	Under condition
C	C	C	C is a constant
x_1	α_1	β_1	
x_1+x_2	$\alpha_1+\alpha_2$	$\beta_1+\beta_2$	
$-x_1$	$-\beta_1$	$-\alpha_1$	
$1/x_1$	$1/\beta_1$	$1/\alpha_1$	$\alpha_1,\beta_1>0$
$x_1\cdot x_2$	$\min(\alpha_1\alpha_2,\alpha_1\beta_2,$ $\alpha_2\beta_1,\alpha_2\beta_2)$	$\max(\alpha_1\alpha_2,\alpha_1\beta_2,$ $\alpha_2\beta_1,\alpha_2\beta_2)$	
$\min(x_1,x_2)$	$\min(\alpha_1,\alpha_2)$	$\min(\beta_1,\beta_2)$	
$\max(x_1,x_2)$	$\max(\alpha_1,\alpha_2)$	$\max(\beta_1,\beta_2)$	

In Table 1.1 Δ is $[\alpha_1,\beta_1]$ or $[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]$. Let us now denote by

$$Term(R,x_1,\dots,x_n,+,\cdot,-,\sqrt[2k+1]{\ },\exp,\sin,\cos,\min,\max) \tag{1.1}$$

the set of all terms built up from the variables $x_1,\dots,x_n$, simbols of some real numbers and functional symbols $+,\cdot,-,\sqrt[2k+1]{\ }$, exp,sin,cos,min,max, where $k>0$ may be any natural number. Suppose that f is a function defined by some term in the set (1.1). How can we determine an m-M pair of f? For instance, how to find an m-M pair of the real function: $x\to\sin(x)$? In order to achieve this we shall apply this:

Proposition 1.2. Let $f:[\alpha_1,\beta_1]\to R$ be a function for which equality

$$f(x)=g(x)-h(x)\qquad (x\in[a_1,b_1])$$

holds, where the functions g,h are continuous and nondecreasing. Then an m-M pair $\langle m(f)(\Delta),M(f)(\Delta)\rangle$ (with $\Delta=[\alpha_1,\beta_1]\subset[a_1,b_1]$) is determined by these equalities: $m(f)(\Delta)=g(\alpha_1)-h(\beta_1)$, $M(f)(\Delta)=g(\beta_1)-h(\alpha_1)$

By Proposition 1.2 and the equality $\sin(x)=(x+\sin(x))-x$ we have the following results $m(\sin)[\alpha_1,\beta_1]=\alpha_1+\sin(\alpha_1)-\beta_1;\quad M(\sin)[\alpha_1,\beta_1]=\beta_1+\sin(\beta_1)-\alpha_1$

Now we pass to the next basic idea. Namely, suppose that f, g are some given m-M functions. Can we find m-M pairs for composed functions like $f+g$, $\sin(f)$, $\max(f,g)$ and so on? The answer is *yes*, and the following lemma 'speaks' of it:

Lemma 1.1. Let the functions $f,h_1,\dots,h_k:D\to R$ and

$$G:\ [A_1,B_1]\times\ldots\times[A_k,B_k]\to R\qquad (A_i,B_i \text{ are given reals})$$

satisfy the following equality

$$f(X) = G(h_1(X),\dots,h_k(X)) \quad (\text{for all } X \in D)$$

Suppose also that for all segments $\Delta \subseteq D$ the inequalities

$$A_i \le m(h_i)(\Delta), M(h_i)(\Delta) \le B_i \quad (i = 1,\dots,k) \tag{*}$$

hold. Then one m-M pair of the function f is defined by the equalities

$$\begin{aligned} m(f)(\Delta) &= (mG)(m(h_1)(\Delta), M(h_1)(\Delta),\dots,m(h_k)(\Delta), M(h_k)(\Delta)) \\ M(f)(\Delta) &= (MG)(m(h_1)(\Delta), M(h_1)(\Delta),\dots,m(h_k)(\Delta), M(h_k)(\Delta)) \end{aligned} \tag{**}$$

providing that all m-M pairs of the functions $h_1,\dots,h_k$, G occuring on the right hand side of these equalities are known.

Proof is ommited (see the proof of Lemma 1.3 in [1]). According to this lemma and Table 1.1 in the m-M Calculus we introduce the following recursive definition:

Definition 1.1.

(i) $m(C)(\Delta) = C$, $M(C)(\Delta) = C$, (C is a constant)
$m(x_i)(\Delta) = \alpha_i$, $M(x_i)(\Delta) = \beta_i$, $(i = 1,\dots,n)$

(ii) $m(f+g)(\Delta) = m(f)(\Delta) + m(g)(\Delta)$, $M(f+g)(\Delta) = M(f)(\Delta) + M(g)(\Delta)$,

(iii) $m(-f)(\Delta) = -M(f)(\Delta)$, $M(-f)(\Delta) = -m(f)(\Delta)$,

(iv) $m(f \cdot g)(\Delta) = \min(m(f)(\Delta) + m(g)(\Delta), m(f)(\Delta)M(g)(\Delta),$
$M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$
$M(f \cdot g)(\Delta) = \max(m(f)(\Delta) + m(g)(\Delta), m(f)(\Delta)M(g)(\Delta),$
$M(f)(\Delta)m(g)(\Delta), M(f)(\Delta)M(g)(\Delta))$

(v) $m(\min(f,g))(\Delta) = \min(m(f)(\Delta), m(g)(\Delta))$
$M(\min(f,g))(\Delta) = \min(M(f)(\Delta), M(g)(\Delta))$

(vi) $m(\max(f,g))(\Delta) = \max(m(f)(\Delta), m(g)(\Delta))$
$M(\max(f,g))(\Delta) = \max(M(f)(\Delta), M(g)(\Delta))$

(vii) $m(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{m(f)(\Delta)}$, $M(\sqrt[2k+1]{f})(\Delta) = \sqrt[2k+1]{M(f)(\Delta)}$

(viii) $m(\exp f)(\Delta) = \exp m(f)(\Delta)$, $M(\exp f)(\Delta) = \exp M(f)(\Delta)$

(ix) $m(\sin f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \sin m(f)(\Delta)$
$M(\sin f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \sin M(f)(\Delta)$

(x) $m(\cos f)(\Delta) = m(f)(\Delta) - M(f)(\Delta) + \cos m(f)(\Delta)$
$M(\cos f)(\Delta) = M(f)(\Delta) - m(f)(\Delta) + \cos M(f)(\Delta)$

An immediate consequence of this lemma is the following:

Lemma 1.2. Let $R^n \to R$ be a function defined by a term $f(x_1,\dots,x_n)$ belonging to set (1.1). This function is an m-M function. Employing Definition 1.1 one of its m-M pairs can be effectively found in a finite number of steps.

In m-M calculus we shall frequently be concerned with dividing some given segments of reals into certain smaller "pieces". In connection with this we introduce the so-called cell-decomposition of a given segment $[a,b] \subset R$. Any such decomposition $\mathcal{D}$ is an infinite set of certain segments $[a',b'] \subseteq [a,b]$, the so-called decomposition cells, where to each cell one of the numbers 0, 1, 2, ..., the so-called **order** of the decomposition, is described. In addition the following conditions are supposed.

Condition 1.1.

(i) $[a,b] \in \mathcal{D}$

(ii) For each[1] $r \in N$ there exists a finite number of cells in $\mathcal{D}$ of order r. The segment $[a,b]$ is the unique cell of order 0.

(iii) The union of all cells of order r is $[a,b]$.

(iv) The interiors of two defferent cells of the same order r are disjoint.

(v) If $d(r)$ denotes the maximum of length of all cells of order r the equality $\lim_{r\to\infty} d(r) = 0$ holds.

A cell-decomposition $\mathcal{D}$ is called a cell-tree if the following condition is fullfiled:

(vi) For each cell $C_r \in \mathcal{D}$ of order $r(>0)$ there exists a unique cell $C_{r-1} \in \mathcal{D}$ of order $r-1$ such that $C_r \subseteq C_{r-1}$.

One example of cell-tree is the so-called **dyadic** tree. Its cells of order r are segments $[\alpha,\beta] \subseteq [a,b]$ defined by the equalities of the form

$$\alpha = a + k(b-a)\cdot 2^{-r}, \quad \beta = \alpha + k(b-a)\cdot 2^{-r}$$

where k can be any element of the set $\{0,1,\ldots,2^r-1\}$. Notice that by the definition of cell-decomposition for each decomposition $\mathcal{D}$ of the segment $[a,b]$ the following fact holds:

Proposition 1.3. To each point $x \in [a,b]$ at least one sequence $\langle C_r(x)\rangle$ of r-cells[2] is related such that the following condition $(\forall r \in N) x \in C_r(x)$, is satisfied.

Any such sequence is called a **cell sequence** of x.

Definition 1.2. 1^0. Let $\mathcal{D}\,[a,b]$ be a cell-decomposition of the segment $[a,b] \subseteq \mathrm{R}$. Then the set of all r-cells of the decomposition is denoted by $\mathcal{D}_r\,[a,b]$.

Let $D = [a_1,b_1] \times \ldots \times [a_n,b_n] \subset R^n$ be an n-dimensional segment and let $\mathcal{D}\,[a_i,b_i]$ be some cell-decompositions of the segments $[a_i,b_i]$ $(i=1,\ldots,n)$. Then the sets $\mathcal{D}_r(D)$, $\mathcal{D}(D)$ (r is a fixed element of N) are introduced respectively by the following equalities:

$$\mathcal{D}_r(D) = \{P_1 \times \ldots \times P_n \mid P_1 \in \mathcal{D}_r[a_1,b_1], \ldots, P_n \in \mathcal{D}_r[a_n,b_n]\}, \quad \mathcal{D}(D) = \cup_{r\in N} \mathcal{D}_r(D)$$

[1] N is the set of all nonnegative integers 0, 1, 2,...

[2] Instead of "cell of order r" we say briefly "r-cell".

Notice that the set (1.1) can be extended to the set

$$Term(R, x_1, \ldots, x_n, +, \cdot, -, \text{ exp,sin,cos,min,max, 1/, arcsin, ln, } \sqrt[k]{\ }) \tag{1.2}$$

which contains new functional symbols: 1/, arcsin, ln, $\sqrt[k]{\ }$, $(k > 1, k \in N)$.
Under some conditions one can find an m-M pair for a function f defined by a term belonging to set (1.2) (see Definitions: 1.3, 1.4, 1.5 and Theorem 1.1 in [1]).

Now we are going to give some formulas for m-M pairs in case of differentiable functions (and complex regular functions).

Theorem 1.1. Let $f : [a_1, b_1] \to R$ be a given function belonging to the class $C^{k+1}[a_1, b_1]$ where k is some natural number. Suppose also that for any segment $\Delta = [\alpha_1, \beta_1]$ (with $\Delta \subseteq [a_1, b_1]$)

$$B(| f^{(k+1)} |)(\Delta) \tag{1.3}$$

denotes an upper bound of the modules of the $(k+1)$-derivative[3] of f when $x \in \Delta$.
Additionally suppose that the following condition is satisfied

$$(\forall \varepsilon > 0)\,(\exists K \in R)\,(diam\,\Delta < \varepsilon \Rightarrow B(| f^{(k+1)} |)(\Delta) < \mathrm{K}) \tag{1.4}$$

that is (1.3) is bounded if $diam\,\Delta \to 0$. Then, using notations $\gamma = \frac{\alpha_1 + \beta_1}{2}$, $\rho = \frac{\beta_1 - \alpha_1}{2}$ one m-M pair of the function f is determined by the following equalities

$$\begin{aligned} m(f)(\Delta) &= f(\gamma) - \sum_{i=1}^{k} \frac{| f^{(i)}(\gamma) |}{i!} \rho^i - \frac{\rho^{k+1}}{(k+1)!} B(|f^{(k+1)}|)(\Delta) \\ M(f)(\Delta) &= f(\gamma) + \sum_{i=1}^{k} \frac{| f^{(i)}(\gamma) |}{i!} \rho^i - \frac{\rho^{k+1}}{(k+1)!} B(|f^{(k+1)}|)(\Delta) \end{aligned} \tag{1.5}$$

Proof is omitted (see [1]).

Definition 1.3. The m-M pair defined by (1.5) is called k-Taylor m-M pair of the function f.

It is interesting that the formulas (1.5) can be generalized to the case of real functions in several variables (see (1.20) in [1]) and also to the case of complex regular functions. Namely, let $f : D \to C$, $D = [a_1, b_1] \times [a_2, b_2]$ be a given complex regular function, k some natural number. Further, let

$$\Delta = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2], \quad \gamma = \frac{\alpha_1 + \alpha_2}{2} + i\frac{\beta_1 + \beta_2}{2}, \quad 2\rho = ((\beta_1 - \alpha_1)^2 + (\beta_2 - \alpha_2)^2)^{1/2}$$

[3] It suffices that x belongs to the interior of Δ only.

Then similarly to (1.5) one k-Taylor m-M pair of the function $|f(z)|$ can be determined by the following equalities

$$\begin{aligned} m(|f|)(\Delta) &= |f(\gamma)| - \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^{i} - \frac{\rho^{k+1}}{(k+1)!}\mathrm{B}_{k+1} \\ M(|f|)(\Delta) &= |f(\gamma)| + \sum_{i=1}^{k} \frac{|f^{(i)}(\gamma)|}{i!}\rho^{i} + \frac{\rho^{k+1}}{(k+1)!}\mathrm{B}_{k+1} \end{aligned} \tag{1.6}$$

where B_{k+1} denotes an upper bound $B(|f^{(k+1)}|)(\Delta)$ supposing that the condition of the type (1.4) is satisfied.

Now we are going to state some applications of the notion of m-M function. First, we will show how a given system of inequalities (particularly equations) can be solved. So, let $D = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset R^n$ be given n-dimensional segment and let $f_i : D \to R (i = 1, \ldots, k)$ be given m-M functions. In connection with them we consider the following system of inequalities

$$f_1(x_1, \ldots, x_n) \geq 0, \ldots, f_k(x_1, \ldots, x_n) \geq 0, \quad (\text{assuming } (x_1, \ldots, x_n) \in D) \tag{1.7}$$

Denote by S the set of all its solutions. In order to determine the set S we shall start with some cell-decomposition $\mathcal{D}(D)$ (see Definition 1.2.). Assume for a moment that $\Delta = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n]$ is any n-dimensional subsegment of D. Generally such a segment can satisfy just one of the following conditions [4]

$$1^0\ (\forall i) M(f_i)(\Delta) \geq 0, \qquad 2^0\ (\exists i) M(f_i)(\Delta) < 0$$

Obviously a segment Δ satisfying condition 2^0 cannot contain any solution of the system (1.7). Accordingly, we introduce the following definition.

Definition 1.4. An n-dimensional segment $\Delta \subseteq D$ is **feasible** in the sense of system (1.7) if the following condition $(\forall i) M(f_i)(\Delta) \geq 0$ is satisfied.

To this definition we add the following obvious remark.

Remark 1.1. If an n-dimensional segment $\Delta \subseteq D$ contains a solution of system (1.7) than Δ must be a feasible segment.

In connection with $\mathcal{D}(D)$ for a fixed $r \in N$ denote by F_r the union of all feasible segments belonging to $\mathcal{D}_r(D)$. Then about system (1.7) we have:

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[4] Instead of $M(f_1)(\Delta) \geq 0, \ldots, M(f_k)(\Delta) \geq 0$, $M(f_1)(\Delta) < 0$ or ... or $M(f_k)(\Delta) < 0$ we have written $(\forall i) M(f_i)(\Delta) \geq 0$, $(\exists i) M(f_i)(\Delta) < 0$ respectively.

Theorem 1.2. The equality $S = \bigcap_{r \in N} F_r$ is true.

Proof is omitted (see the proof of Theorem 2.1 in [1]).

Obviously Theorem 1.2 suggests an idea how to solve system (1.7), briefly said how to find F_r step-by-step. To improve such a procedure we can define the set F_{r+1} as a subset of the set F_r. This idea is used in the following solving procedure for system (1.7):

Procedure 1.1. Solving procedure depends on a cell-decomposition $\mathcal{D}(D)$. If we want it to be a cell-tree then we can choose it in advance. Otherwise we determine $\mathcal{D}_r(D)$ during the solving procedure. Further, step-by-step we form a sequence $\langle F'_r \rangle$ whose each member F'_r is the union of some feasible products $P_r \in \mathcal{D}_r(D)$. This sequence is defined inductively as follows:

1^0 $F'_0 = D$ if D is feasible, otherwise $F'_0 = \emptyset$.

2^0 For any $r \in N$: F'_{r+1} = The union of all products $P_{r+1} \in \mathcal{D}_{r+1}(D)$ such that P_{r+1} is feasible and $P_{r+1} \subseteq F'_r$.

If $\mathcal{D}(D)$ is a cell-tree then condition $P_{r+1} \subseteq F'_r$ is satisfied, according to the definition of a cell-tree. Otherwise, we should define $\mathcal{D}_{r+1}[a_1, b_1], \ldots, \mathcal{D}_{r+1}[a_n, b_n]$ in the $(r+1)^{-th}$ step so that the condition $P_{r+1} \subseteq F'_r$ is satisfied.

If for some $r \in N$ we have the equality $F'_0 = \emptyset$ then the procedure halts and S is $\emptyset$. Otherwise, the sets F'_r when r is getting greater and greater give better and better approximations of the set S.

Notice that the sequences $\langle F_r \rangle$, $\langle F'_r \rangle$ may differ but nonetheless the equality $\bigcap_{r \in N} F_r = \bigcap_{r \in N} F'_r$ always holds. Besides that the sequences $\langle F'_r \rangle$ is monotone, for the inclusions $F'_0 \supseteq \ldots \supseteq F'_r \supseteq F'_{r+1} \supseteq \ldots$ are satisfied. In general about the nature of the procedure one may say the following:

Proposition 1.4. Using the fact that non feasible cells cannot contain any solution we actually reject step-by-step various solution-free "pieces" of the given domain D. Additionally, the non-feasibility criterion is so fine that every point $(x_1, \ldots, x_n) \in D$, which is not a solution, will be rejected at some step r. Accordingly if system (1.7) has no solutions then at some step r all products $P_r \subseteq F'_r$ will be non-feasible, which implies the conclusion $S = \emptyset$.

About system (1.7) we also add the following. For some products P_r it may happen that all inequalities $m(f_1)(P_r) \geq 0, \ldots, m(f_k)(P_r) \geq 0$ are satisfied. Obviously such products must be subsets of the set S. Consequently we have the following definition.

Definition 1.5. An n-dimensional segment $\Delta \subseteq D$ is **a solutional segment** in the sense of system (1.7) if the condition $(\forall i) m(f_i)(\Delta) \geq 0$ is fulfilled.

Besides this definition, the so-called **indeterminated segments** are defined by the following conditions

$$(\forall i)M(f_i)(\Delta) \geq 0, \quad (\exists i)m(f_i)(\Delta) < 0 \qquad (1.8)$$

In others words, a segment $\Delta \subseteq D$ is indetermined if and only if Δ is a feasible but not a solutional segment. Using the notions of solutional and indetermined products the solving Procedure 1.1 can be profoundly improved as follows.

Procedure 1.2. Step-by-step we form sequences $\langle S_r \rangle$, $\langle U_r \rangle$ whose members S_r, U_r are unions of some solutional, indetermined products P_r respectively. Their inductive definitions reads:

1^0 $S_0 = D$ if D is a solutional product, otherwise $S_0 = \emptyset$
$U_0 = D$ if D is an indetermined product, otherwise $U_0 = \emptyset$

2^0 For any $r \in N$
$S_{r+1} = S_r \cup$ The union of all solutional products $P_{r+1} \subseteq P_r$
$U_{r+1} =$ The union of all indetermined products $P_{r+1} \subseteq P_r$

If for some $r \in N$ we obtain the equality $S_r \cup U_r = \emptyset$ then the procedure halts and the equality $S = \emptyset$ is true. Similarly, if for some $r \in N$ $U_r = \emptyset$ then the procedure halts too and the equality $S = S_r$ is true.

Otherwise, i.e. if for every $r \in N$ both relations $S_r \cup U_r \neq \emptyset$, $U_r \neq \emptyset$ are fulfilled, the sets $S_r \cup U_r$ when r is getting greater and greater give better and better approximations of the set S.

Remark that the sequences $\langle S_r \rangle$, $\langle U_r \rangle$ have the following properties:

$$S_r \subseteq S_{r+1}, \quad U_r \subseteq U_{r+1}, \quad F_r' = S_r \cup U_r, \quad (r = 0,1,\dots)$$

Notice that one particular case of (1.7) is when it is a system of equations, like

$$f_1(x_1,\dots,x_n) = 0,\dots,f_k(x_1,\dots,x_n) = 0 \qquad (1.9)$$

In such a case Definition 1.4 reduces to the following feasibility-definition:

Definition 1.6. A n-dimensional segment $\Delta \subseteq D$ is a feasible set in the case of system (1.9) if the condition

$$(\forall i)(m(f_i)(\Delta) \leq 0 \leq M(f_i)(\Delta)) \qquad (1.10)$$

is satisfied.

Let now $f : D \to C$ (dim $D = 2$) be a given complex function and suppose that at least one effective formula for $m(|\ f\ |)$ is known (see (1.6)). Then solving the equation $f(z) = 0$ in $z \in D$ may be treated as solving the real equation $|f(z)| = 0$. Now the definition of feasible segment $\Delta \subseteq D$ reads:

A segment $\Delta \subseteq D$ is feasible in the sense of equation $|f(z)| = 0$ (1.11)
if the condition $m(|\ f\ |)(\Delta) \le 0$ holds.

An important particular case is provided when f is a polynomial function determined, say, by

$$f(z) = a_n z^n + \cdots + a_0 \quad (a_n \neq 0)$$

where $a_n, \ldots, a_0$ are given complex numbers. Then a domain $D = [-r,r] \times [-r,r]$ which contains all solutions of the equation $f(z) = 0$ can be effectively found. For example, by Cauchy's formula for the number r we can take

$$r = 1 + \max_{0 \le i \le n-1} (|a_i| / |a_n|) \tag{1.12}$$

Now, we give some concrete examples. We emphasize that generally $fis(r)$ will denote the number of all feasible products of r-cells.

Example 1.1. Equation $\sin x = 1/x$, $x \in [1, 20]$.

Let $[\alpha, \beta] \subseteq [1, 20]$ be any segment. Then according to Definition 1.1. (ix) for the function $f(x) = \sin x - 1/x$ one m-M pair is defined by

$$m(f)[\alpha, \beta] = \alpha + \sin\alpha - \beta - 1/\alpha, \quad M(f)[\alpha, \beta] = \beta + \sin\beta - \alpha - 1/\beta$$

By Procedure 1.1, using the feasibility Definition 1.6 and dyadic tree, the number $fis(r)$ of all feasible r-cells step-by-step up to $r = 25$ is given in the following list (its elements have the form (step r, $fis(r)$)).

(1,1), (2,2), (3,4), (4,8), (5,15), (6,16), (7,16), (8,15), (9,16), (10,14),
(11,14), (12,16), (13,16), (14,15), (15,15), (16,15), (17,17), (18,16),
(19,15), (20,15), (21,15), (22,15), (23,15), (24,15), (25,15)

As we see starting with the step 5 the number of all feasible cells $fis(r)$ is about 16. Consequently, according to Procedure 1.1, in these steps we should test about $16 \cdot 2$ (i.e. 32) cells only. For instance, exactly said, in the 20th step there are all together 2^{20} cells, but we should test only 30 of them. In the step 25 we obtain the following numerical result:

The given equation has 7 solutions described as follows

$1.11415595 \le x_1 \le 1.11415821$; $2.77260345 \le x_2 \le 2.77260572$
$6.4391157 \le x_3 \le 6.4391191$; $9.31724286 \le x_4 \le 9.31724399$
$12.6455307 \le x_5 \le 12.6455341$; $15.6439972 \le x_6 \le 15.6439983$
$18.9024819 \le x_7 \le 18.9024853$.

Example 1.2. Complex equation in $z = x + iy$

$$z^8 + (A_7 + iB_7)z^7 + \ldots + (A_0 + iB_0) = 0$$

where $A_7, B_7, \ldots, A_0, B_0$ are given real numbers.

All solutions lie in the domain $[-r,r] \times [-r,r]$ where r is defined by (1.12). Using Procedure 1.1, definition of type (1.11) and dyatic trees, several equations are solved up to 25$^{\text{th}}$ step. In all of them the coefficients were chosen at random. It is interesting, that the numbers $fis(r)$, when $r \ge 6$ are pretty small. Namely, in the 25$^{\text{th}}$ step this number is always less then 15. We give concrete numerical results in the case when coefficients A_j, B_j are determined as follows

$A_7 = -0.628871968$ $B_7 = -0.90620273$
$A_6 = 0.655487601$ $B_6 = 0.109498452$
$A_5 = 0.794467662$ $B_5 = 0.145832495$
$A_4 = 0.677786328$ $B_4 = 0.862459254$
$A_3 = -0.623235982$ $B_3 = 0.945879881$
$A_2 = 0.552867495$ $B_2 = -0.164039785$
$A_1 = 0.658555102$ $B_1 = 0.618662189$
$A_0 = 0.934256145$ $B_0 = 0.147878684$

The solutions $x_j + iy_j \quad (j = 1,\ldots,8)$ are described as follows

$-0.340724289 \le x_1 \le -0.340724140$, $-0.793053508 \le y_1 \le -0.793053359$
$-0.897440463 \le x_2 \le -0.897440314$, $-0.308772177 \le y_2 \le -0.308772027$
$0.385927558 \le x_3 \le 0.3859277070$, $-0.954408497 \le y_3 \le -0.954408199$
$1.117326470 \le x_4 \le 1.1173266200$, $-0.460661650 \le y_4 \le -0.460661501$
$-0.310650319 \le x_5 \le -0.310650170$, $0.521920323 \le y_5 \le 0.521920621$
$-0.707707405 \le x_6 \le -0.707707107$, $0.786857605 \le y_6 \le 0.786857754$
$0.643312931 \le x_7 \le 0.643313080$, $0.492128873 \le y_7 \le 0.492128879$
$0.738826841 \le x_8 \le 0.738826990$, $1.622191220 \le y_8 \le 1.622191370$

Example 1.3. Complex equation in z: $e^x = z$

In the domain $[-20, 20] \times [-20,20]$ this equation has 6 solutions $x_j + iy_j \quad (j = 1,\ldots,6)$ described as follows

$$2.653191109 \le x_1 \le 2.65319228, \qquad -13.94920826 \le y_1 \le -13.94920731$$
$$2.062276600 \le x_2 \le 2.06227899, \qquad -7.58863215 \le y_2 \le -7.58863020$$
$$0.318130250 \le x_3 \le 0.31813264, \qquad -1.33723736 \le y_3 \le -1.33723497$$

$$x_4 + iy_4 = x_3 - iy_3, \quad x_5 + iy_5 = x_2 - iy_2, \quad x_6 + iy_6 = x_1 - iy_1$$

The calculations were done up to the 25th step (bisection way). Starting with the 6th step the number $fis(r)$ is about 16. For instance: $fis(24) = 15$, $fis(25) = 16$.

Example 1.4. We consider the system in $(x,y,z) \in D \subset R^3$

$$e^x + x + \sin y + \cos z = p$$
$$x^3 + e^{\sin y} - z - e^z = q$$
$$\sin(x - z) + (x + y)^5 - x - y - z = r \qquad (p,q,r \text{ are given real numbers})$$

Notice that in all cases stated below again dyadic trees are used.

Case 1: $p = 2, q = 0, r = 0, D = [1,2]\times[-2,1]\times[-3,2]$. There is exactly one solution $(x,y,z) = (0,0,0)$. Starting with the 6th step the number $fis(r)$ was between 40 and 50. In the 24th step we obtained the following result

$$-0.00001525878910 \le x \le 0.0000247955322$$
$$-0.00002479553220 \le y \le 0.0000324249268$$
$$-0.00000762939453 \le z \le 0.0000114440918$$

Case 2: $p = 2, q = 0, r = 0, D = [-5,5]\times[1,5]$. Step-by-step the number $fis(r)$ is 1, 8, 21, 32, 24, 0. Therefore, we conclude that the system has no solution.

Remark 1.2. This example illustrates one of the key features of the m-M calculus generally:

> If some problem has no solution in a given domain D then there exists a step k such that $fis(k) = 0$.

In order words, the non-existence of solutions can be positively established at some step k.

We point out that part 4. of the Section 2: **System of equations, system of inequalities** in [1] deals with the feasibility problem.

Now we move on to other applications of m-M functions. So, in the Section 3, of [1] it is stated how, under some conditions, one can approximately calculate a given n-dimensional Riemann integral. We take Example 3.1 from [1]; here this is:

Example 1.5. For the integral I: $\iint_{Cond(x,y)} xydxdy$ where $Cond(x,y)$ reads

$$0 \le x \le 2,\ 0 \le y \le 2,\ \ 2 + e \cdot (x + y + 2) \ge e^{x+1} + e^{y+1}$$

by using 4-tree we have the following results

Step 1:	$0.0000000000000000 \le I \le 0.5625000000000000$
Step 2:	$0.0278320312500000 \le I \le 0.2424316406250000$
Step 3:	$0.0950307846069336 \le I \le 0.1581497192382812$
Step 4:	$0.1179245151579380 \le I \le 0.1341890022158623$
Step 5:	$0.1239582093403442 \le I \le 0.1281216432544170$
Step 6:	$0.1255188694198068 \le I \le 0.1265613023800256$
Step 7:	$0.1259090350460332 \le I \le 0.1261702301487544$
Step 8:	$0.1259090350460332 \le I \le 0.1259090350460332$
Step 9:	$0.1259090350460332 \le I \le 0.1259090350460332$

But, what would happen if $Cond(x,y)$ has no solution in x,y? We emphasize that in this case such a fact can be established at some step k of the calculating procedure.

Further, in the Section 3 of [1] it is stated how, under some conditions, one can find an m-M pair of function defined by some integral (see (3.9) in [1]) or by some infinite sum (see (3.11) in [1]). In connection with this fact we state an example (this is Example 3.2 in [1]):

Example 1.6. Let f be a function defined by

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{x + 2^i}$$

Concerning the equation $f(x) = c$ with $a \le x \le b$ where a, b, c are constants we have the following results (dyadic tree is used).

Case $c = 1.5,\ a = 0,\ b = 1$. In the step 20 we obtain the following double inequality

$$0.54416 \le x \le 0.54417$$

The numbers $fis(r)$ $(r = 1,2,\ldots,20)$ are in turn

1, 2, 3, 3, 2, 3, 2, 2, 3, 2, 3, 3, 2, 3, 3, 2, 2, 2, 3, 2, 3, 2, 3

Case $c = 1.5,\ a = 0.6,\ b = 1$.

Step 1: $fis(1) = 1$; Step 2: $fis(2) = 1$; Step 3: $fis(3) = 0$.
Conclusion: $f(x) = c$ has no solutions.

Similarly, concerning the equation $f(x) = x$ with $a \le x \le b$ we have the following results

Case $a = 0, b = 1$. In the third step $fis(r) = 0$, so the given equation has no solutions.

Case $a = 0, b = 2$. The $fis(r)$ $(r = 1,2,\ldots,20)$ is 1 or 2. In the step 20 we obtain the following double inequality $1.19055 \le x \le 1.190056$.

2. m-M PAIRS OF THE FIRST ORDER <, ≤ FORMULAS. APPLICATIONS

This is the crucial part of m-M Calculus. In the part we use the notion *of the first order* $<,\le$ *formulas*. Briefly said[5], these are the formulas built up from some variables, the symbols of real numbers, symbols of some m-M functions $f, g, \ldots$, the relational symbols $<,\le$ and finally the logical symbols $\wedge,\vee,\neg,\forall,\exists$. For instance,

$$f(x) < g(x,y), \quad (\sin(f(x)) \le g(x) \wedge g(y) \le f(x)) \vee f(y) < g(x),$$
$$(\forall x \in [a,b]) f(x) \le f(y), \quad (\forall x \in [a,b])(\exists y \in [c,d]) f(x,y) \le 2$$

are examples of such formulas. Since the quantifiers $\forall,\exists$ may occur in such formulas, we introduce the notions of the **free** and **bound** variable. A variable, say v, is free in some formula ϕ if v does not occur in some part of ϕ which has a form $(\forall v)(\ldots)$ or a form $(\exists v)(\ldots)$ where (...) denotes the scope of the quantifier. For instance, x, y are free variables in the formula $(\exists z) g(x,y) < g(x,z)$, while z is not free in it. A variable v is bound if it is not free. So, in the last formula z is a bound variable.

As we know from Section 1 if we want to solve an inequality like

$$f(x) \le 0 \qquad \text{(where } x \in [a,b]) \tag{*1}$$

then supposing that we use some cell-decomposition the basic idea is to use the following implication

$$f(x) \le 0 \Rightarrow m(f)(C_r(x)) \le 0 \qquad \text{(where } r = 0,1,2,\ldots) \tag{*2}$$

Namely, according to (*2) we have had the definition of the feasible cell Δ: Δ is feasible $\leftrightarrow m(f)(\Delta) \le 0$. Clearly, if Δ is not feasible then Δ contains no solution of (*1). But, besides that we also have the following implication

$$M(f)(C_r(x)) \le 0 \Rightarrow f(x) \le 0 \tag{*3}$$

which yields the following fact: if for some Δ we have the inequality $M(f)(\Delta) \le 0$ then any element of such a Δ is a solution of (*1), i.e. this Δ is '**a solutional**' cell. Both implications (*2), (*3) can be written in this way:

$$M(f)(C_r(x)) \le 0 \Rightarrow f(x) \le 0 \Rightarrow m(f)(C_r(x)) \le 0 \quad r = 0,1,2,\ldots \tag{*4}$$

[5] For details see (4.1), (4.2) in [1].

Until now we have used the notation of the m-M pair, which is deeply related to the ordinary notions of minimum and maximum. Besides that in m-M calculus we use a more subtle notion; namely we introduce the so-called $m(\phi)$ and $M(\phi)$ where ϕ may be a first order $<, \leq$ formula. For instance, if ϕ is the formula $f(x) \leq 0$ occuring in (*4) then the left hand and right hand side in (*4), i.e. the formulas $M(f)(C_r(x)) \leq 0$ and $m(f)(C_r(x)) \leq 0$ are $M(\phi)$ and $m(\phi)$ respectively. As a matter of fact, in order to emphasize the number r and the cell $C_r(x)$ instead of $M(\phi)$, $m(\phi)$ we shall use the following denotations $M_r(\phi)(C_r(x))$ and $m_r(\phi)(C_r(x))$ respectively. Using these denotations (*4) can be rewritten in the following way[6]

$$M_r(\phi)(C_r(x)) \Rightarrow \phi(x) \Rightarrow m_r(\phi)(C_r(x)) \qquad r = 0,1,2,\ldots \tag{*5}$$

As we shall in the sequel explain in general we have the following two facts:

First, for any first order $<, \leq$ formula $\phi(x_1,\ldots,x_m)$ whose all free variables (2.1) are among $x_1,\ldots,x_m$ one can effectively determine its

$$M_r(\phi)(C_r(x_1),\ldots,C_r(x_m)), \quad m_r(\phi)(C_r(x_1),\ldots,C_r(x_m)).$$

Second, like (*5) the following double implication

$$M_r(\phi)(C_r(x_1),\ldots,C_r(x_m)) \Rightarrow \phi(x_1,\ldots,x_m) \Rightarrow m_r(\phi)(C_r(x_1),\ldots,C_r(x_m)) \tag{2.2}$$

holds.

Notice that fact (2.1) is contained in Definition 4.1 from [1]. However, this definition is rather technically complex. Here we shall explain (2.1) by means of some examples:

(i) Formula $\phi(x)$ is $f(x) < g(x)$. Then

$$\begin{aligned} m_r(\phi)(C_r(x)) &:= m(f)(C_r(x)) < M(g)(C_r(x)) \\ M_r(\phi)(C_r(x)) &:= M(f)(C_r(x)) < m(g)(C_r(x)) \end{aligned}$$

The double implication of the form (2.2), i.e. the implication

$$M(f)(C_r(x)) < m(g)(C_r(x)) \Rightarrow f(x) < g(x) \Rightarrow m(f)(C_r(x)) < M(g)(C_r(x))$$

is obviously true. Generally if ϕ is a formula of the form $P < Q$ then:

$$m_r(\phi)(\ldots) := m(P)(\ldots) < M(Q)(\ldots) \quad M_r(\phi)(\ldots) := M(P)(\ldots) < m(Q)(\ldots)$$

where the symbols ... stand for omitted arguments. The double implication of the form (2.2) is true. We remark that in the previous examples the simbol $<$ may be replaced by $\leq$.

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[6] Instead ϕ we wrote $\phi(x)$ to point out that ϕ has a free variable x.

(ii) Formula ϕ is a conjuction of the form $\alpha \wedge \beta$. Then (according to part (ii) in Definition 4.1) we have:

$$m_r(\phi)(\ldots) := m_r(\alpha) \wedge m_r(\beta) \quad M_r(\phi)(\ldots) := M_r(\alpha) \wedge M_r(\beta)$$

For instance if ϕ is a formula $f(x) < 0 \wedge g(x) < 0$ then:

$$m_r(\phi)(C_r(x)) := m(f)(C_r(x)) < 0 \wedge m(g)(C_r(x)) < 0$$
$$M_r(\phi)(C_r(x)) := M(f)(C_r(x)) < 0 \wedge M(g)(C_r(x)) < 0$$

We remark that in this example the symbol $\wedge$ may be replaced by $\vee$ (which is related to part (iii) in Definition 4.1)).

(iii) Formula ϕ is $(\exists x \in [a,b]) f(x) < g(x)$. Then (according to part (v) in Definition 4.1) we have:

$$m_r(\phi) := (\exists X \in \mathcal{D}_r([a,b]))\, m(f)(X) < M(g)(X)$$
$$M_r(\phi) := (\exists X \in \mathcal{D}_r([a,b]))\, M(f)(X) < m(g)(X)$$

where $\mathcal{D}_r([a,b])$ is the set of all r-cells of the segment $[a,b]$. Similarly, if ϕ is a formula $(\forall x \in [a,b]) f(x) < g(x)$ then:

$$m_r(\phi) := (\forall X \in \mathcal{D}_r([a,b]))\, m(f)(X) < M(g)(X)$$
$$M_r(\phi) := (\forall X \in \mathcal{D}_r([a,b]))\, M(f)(X) < m(g)(X)$$

(iv) Formula $\phi(x)$ is $(\forall y \in [a,b]) f(x) \le f(y)$. Then we have:

$$m_r(\phi)(C_r(x)) := (\forall Y \in \mathcal{D}_r([a,b]))\, m(f)(C_r(x)) \le M(g)(Y)$$
$$M_r(\phi)(C_r(x)) := (\forall Y \in \mathcal{D}_r([a,b]))\, M(f)(C_r(x)) \le m(g)(Y)$$

(v) Formula $\phi(z)$ is $(\forall x \in [a,b])(\exists y \in [c,d]) f(x,y,z) < 0$. Then we have:

$$m_r(\phi)(C_r(z)) := (\forall X \in \mathcal{D}_r[a,b])(\exists Y \in \mathcal{D}_r[c,d])\, m(f)(X \times Y \times C_r(z)) < 0$$
$$M_r(\phi)(C_r(z)) := (\forall X \in \mathcal{D}_r[a,b])(\exists Y \in \mathcal{D}_r[c,d])\, M(f)(X \times Y \times C_r(z)) < 0$$

Next, notice that it is not dificult to prove (2.2) (see Theorem 4.1 in [1]).

Now, by means of two examples we shall see how one can practically use double implications (2.2). In both examples the notion of a positive $\le$ – first order formula will appear. Namely, a formula ϕ is $\le$ – positive if none of the symbols $\neg$, $<$ occurs in ϕ.

Example 2.1. Determine all points $x \in [a,b]$ at which a given m-M function f : $[a,b] \to [a,b]$ attains its minimum.

m-M solution. First, formula (iv) above corresponds to this problem. According to (2.2) we have the following implication:

$$\phi(x) \Rightarrow (\forall Y \in \mathcal{D}_r([a,b]))\, m(f)(C_r(x)) \leq M(g)(Y) \qquad (*1)$$

Next, assume that we use some cell-decomposition and $\Delta \subset \mathcal{D}_r([a,b])$ is any r-cell (at some step r). If Δ contains a point a at which f attains its minimum then we can comprehed Δ as $C_r(a)$. Doing this way by (*1) we conclude that the sell Δ must satisfy the following condition:

$$(\forall Y \in \mathcal{D}_r([a,b]))\, m(f)(\Delta) \leq M(g)(Y)$$

Related to this we introduce the notion of a feasible cell Δ:

$\Delta \in \mathcal{D}_r([a,b])$ is feasible if and only if $(\forall Y \in \mathcal{D}_r([a,b]))\, m(f)(\Delta) \leq M(g)(Y)$

Now clearly in order to solve the given problem we can use a procedure which is very similar to Procedure 1.1 and Procedure 1.2. But, obviously the following question appears: Can we by using only the sets of feasible cells obtain the set S of all solutions of the given problem? The answer is *yes*, because a theorem like Theorem 1.2 holds. The basic reason is: formula $\phi(x)$ is $\leq$ - positive (see (2.6) below).

We also add the following remark. We can optimize the described procedure by 'diminishing for loop $(\forall Y \in \mathcal{D}_r([a,b]))$' (see Problem 2.1 below).

Example 2.2. Is there any function $\psi : [a,b] \to [c,d]$ satisfying the following equation $f(x,\psi(x)) = 0$ (for all $x \in [a,b]$) where a, b, c, d are given reals and $f : [a,b] \times [c,d] \to R$ is a given m-M function?

m-M solution. First, the following formula ϕ

$$(\forall x \in [a,b])(\exists y \in [c,d])(f(x,y) \leq 0 \wedge 0 \leq f(x,y))$$

corresponds to the given problem. Clearly, the function ψ exists if and only if the formula ϕ is true. Suppose that we use some cell-decomposition of $[a,b] \times [c,d]$. Then like the example (v) above for formula ϕ we have the following m-M pair:

$$M_r(\phi) := (\forall X \in \mathcal{D}_r([a,b]))(\exists Y \in \mathcal{D}_r([c,d]))\, M(f)(X \times Y) \leq 0 \wedge 0 \leq m(f)(X \times Y))$$
$$m_r(\phi) := (\forall X \in \mathcal{D}_r([a,b]))(\exists Y \in \mathcal{D}_r([c,d]))\, m(f)(X \times Y) \leq 0 \wedge 0 \leq M(f)(X \times Y))$$

Then the double implication (2.2) reads: $M_r(\phi) \Rightarrow \phi \Rightarrow m_r(\phi)$, where $r = 0,1,\ldots$. According to this implication we state the following procedure:

Procedure 2.1. Set $r := 0$.

(i) Calculate $M_r(\phi)$. If $M_r(\phi)$ is true then go to (iii), else calculate $m_r(\phi)$. If $m_r(\phi)$ is false then go to (ii) else set $r := r+1$ and go (i)

(ii) Procedure stops and the answer is: the function ψ does not exist.

(iii) Procedure stops and the answer is: the function ψ exists.

Concerning this procedure there are two cases:

(j) The procedure stops at some step r.
(jj) The procedure never stops.

In the case (j) we have one of the result (i), (ii).

In the second case, since ϕ is a $\leq$-positive formula the answer is *yes* (see (2.6) below). But, in practice in the second case by performing the Procedure 2.1 up to some 'big' number r we can 'approximatively' solve the given problem.

We also add the following remark:

If r is a fixed natural number then we can effectively calculate $m_r(\phi)$ and $M_r(\phi)$ (2.3)

Indeed, let $\mathcal{D}_r([a,b]) = \{A_1,\dots,A_p\}$, $\mathcal{D}_r([c,d]) = \{C_1,\dots,C_q\}$, where p, q are some constants. Then, for instance, for $m_r(\phi)$ we have the following equality

$$m_r(\phi) = \bigwedge_{i=1}^{p} \bigvee_{j=1}^{q} (m(f)(A_i \times B_j) \leq 0 \wedge 0 \leq M(f)(A_i \times B_j))$$

Now suppose that

Prob is a mathematical problem expressed by some first order $<, \leq$-formulae $\phi(x_1,\dots,x_m)$ $(m \geq 0)$, whose all free variables are among $x_1,\dots,x_m$. It is assumed that each variable v of ϕ has a given corresponding segment. If $m > 0$ let $(x_1,\dots,x_m) \in D$, where $D \subset R^m$ is a given m-segment. Then, to solve **Prob** means: find all values of $x_1,\dots,x_m$ such that the formula $\phi(x_1,\dots,x_m)$ is satisfied. If $m = 0$ then, to solve **Prob** means: establish whether the formula ϕ true. (2.4)

We point out that the class of all problems **Prob** of type (2.4) is very wide. In mathematics, particularly in numerical analysis, there are many problems of type (2.4), For instance, Problems 2.1, 2.2, 2.3, 2.4, 2.5 (below in this section) belong to this class. However, we also emphasize that there are a lot of problems of type (2.4) which until now have not been treated in mathematics but m-M Calculus offers new means to do this.

Now we are going briefly to describe how we can solve a problem of type (2.4). First of all we should use some cell-decomposition. Namely, let all variables involved in ϕ, free or bound, be $v_1,\dots,v_t$. Denote their segments by $I(v_i)$ $(i = 1,\dots,t)$. Suppose that for each of them one cell-decomposition $\mathcal{D}(I(v_i))$ is chosen (see Definition 1.2). We distinguish two cases: $m=0$ and $m>0$. If $m=0$ then according to (2.2) we use Procedure 2.1. If $m>0$ then according to (2.2) we introduce the notions of feasible / solutional n-dimensional segment Δ. This definition generally reads (in fact this is Definition 4.2 in [1]):

Definition 2.1. Let $r \in N$ be a given number and let $P_r = X_1 \times \ldots \times X_m$ be a Cartesian product of some r-cells X_i (with $X_i \in \mathcal{D}_r(I(x_i))$). Then:

(i) The product P_r is a feasible product in the sense of the formula $\varphi(x_1,\ldots,x_m)$ if and only if the condition $m_r(\varphi)(X_1,\ldots,X_m)$ is satisfied.

(ii) The product P_r is a solutional product in the sense of the formula $\varphi(x_1,\ldots,x_m)$ if and only if the condition $M_r(\varphi)(X_1,\ldots,X_m)$ is satisfied.

Having in mind Definition 2.1 and double implication (2.2) one can easily conclude the following double inclusion:

$$\bigcup_{r=0,1,\ldots} S_r \subseteq S(\phi) \subseteq \bigcap_{r=0,1,\ldots} F_r \tag{2.5}$$

where S_r is the union of all solutional products P_r, F_r is the union of all feasible products P_r, and $S(\phi)$ is the set of all solutions of the formula $\varphi(x_1,\ldots,x_m)$(with $x_i \in (I(x_i))$. We point out that for each $r=0,1,\ldots M_r(\varphi)(X_1,\ldots,X_m)$, $m_r(\varphi)(X_1,\ldots,X_m)$ are finite expressions; consequently we can effectively find sets S_r, F_r (see (2.3)). According to this fact we can describe a procedure by which we step-by-step calculate the sets S_r, F_r; in other words in such a way we approximately solve a problem of type (2.4). Concerning (2.5) we emphasize that for some formulas the equality

$$S(\phi)=\bigcap_{r=0,1,\ldots} F_r \tag{2.6}$$

can be true. For instance, this equality is true for positive $\leq$ formulas. Recall, these are the formulas which do not contain symbols $<, \neg$. Similarly, if ϕ is a positive $<$ formula (i.e. does not contain symbols $\leq, \neg$) then we have the following equality

$$S(\phi)=\bigcap_{r=0,1,\ldots} S_r \tag{2.7}$$

Notice that equalities (2.6), (2.7) appear in Theorem 4.4 from [1].

In the sequel we are going to state several problems for which one can use an equality of the form (2.6).

Problem 2.1. Let $f: D \to R$ ($D=[a_1,b_1] \times \ldots \times [a_n,b_n]$) be a given m-M function. We seek all points $(x_1,\ldots,x_n) \in D$ at which this function attains the minimum value, i.e. we solve in $(x_1,\ldots,x_n) \in D$ the following formula $\phi(x_1,\ldots,x_n)$:

$$(\forall y_1 \in [a_1, b_1])\ldots(\forall y_n \in [a_n, b_n])\, f(x_1,\ldots,x_n) \le f(y_1,\ldots,y_n) \tag{*2}$$

The formula (*2) is $\le$ positive, so we can use equality (2.6). Notice that the corresponding procedure is similar to Procedure 1.1 and 1.2.

According to Definition 2.1 a product $X_1 \times \ldots \times X_n$ is feasible in the sense of the formula $\phi(x_1,\ldots,x_n)$ if and only the condition

$$(\forall Y_1 \in \mathcal{D}_r\,[a_1, b_1])\ldots(\forall Y_n \in \mathcal{D}_r\,[a_n, b_n])\; m(f)(X_1 \times \ldots \times X_n) \le M(f)(Y_1 \times \ldots \times Y_n) \tag{*3}$$

is satisfied. This problem additionally has the following particular property:

Seeking the points at which the function f attains minimum in the set D may be replaced by seeking such points in the set F_r, where $r = 0,1,\ldots$ (*4)

This fact can be used in the following manner

In the solving procedure we step-by-step replace the initial domain D by the sets $F_1,\ldots,F_r,\ldots$ respectively. (*5)

Suppose now that

Function f has the first order partial derivatives $f'_{x_1},\ldots,f'_{x_n}$ and these derivatives are m-M functions. (2.8)

Assume we know that function f attains its minimum at some point $(c_1,\ldots,c_n) \in InteriorD$. Then to the feasibility criterion (*3) we may add the following new requirements:

$$m(f'_{x_i})\; X_1 \times \ldots \times X_n \le 0 \le M(f'_{x_i})\; (X_1 \times \ldots \times X_n) \quad \text{where } i = 1,\ldots,n$$

which is related to the fact that the equalities $f'_{x_1} = 0,\ldots,f'_{x_n} = 0$ must be satisfied at the point $(c_1,\ldots,c_n)$. We point out that such an property:

A possibility to add some new requirements to the general feasibility criterion

is one of the nicest features of m-M calculus.

Remark 2.1. In m-M Calculus we usually use 'the cell-decomposition strategy'. But, we can use another strategy as well. Here we shall state sketch[7] of a procedure *LS* by which under some conditions one can find a local minimum or a saddle point of the function *f* from Problem 2.1.

---

[7] A complete version will be published separately.

Let δ be some positive real number, choosen arbitrarily. If $(x_1,...,x_n) \in D$ then by $\Delta(x_1,...,x_n,\delta)$ we denote the Cartesian product $[x_1-\delta, x_1+\delta] \times ... \times [x_n-\delta, x_n+\delta]$. We suppose that $f: D \to R$ satisfies the condition

Function f has the second order partial derivatives $f''_{x_1,x_1},...,f''_{x_n,x_n}$ and these derivatives are m-M functions in each $\Delta(x_1,...,x_n,\delta)$ where $(x_1,...,x_n) \in D$.

In the procedure *LS* we shall use the following general fact:

Let $g: [a-h, a+h] \to R$ be a function having the first order derivative $g'(x)$ for every $x \in (a-h, a+h)$, whose modulus $|g'(x)|$ is bounded by some positive constant K. If $g(a) > 0$ then $g(x) > 0$ for every $x \in [a-h', a+h']$ where $h' = \min(h, g'(a)/K)$

In procedure *LS* we use the following constants, chosen arbitrarily:

$S_{\max}$ - the maximum number of steps in the procedure
Mem $\in \{0,1,...,n\}$ - an auxiliary number.

Procedure *LS* (partly described in 'Pascal style') reads:

We start with an initial point $(p_1,...,p_n)$ from D.

$k := 1$; For $i := 1$ to n do $x_i := p_i$;
100: $Mem := 0$;

For $i := 1$ to n do
Begin
If $f'_{x_i}(x_1,...,x_n) > 0$ then
Begin if $x_i = a_i$ then $Mem := Mem + 1$ else
$x_i := x_i - \min(\delta, f'_{x_i}(x_1,...,x_n) / M(|f''_{x_i,x_i}|)(\Delta))$
End
else $f'_{x_i}(x_1,...,x_n) < 0$ then
Begin if $x_i = b_i$ then $Mem := Mem + 1$ else
$x_i := x_i + \min(\delta, f'_{x_i}(x_1,...,x_n) / M(|f''_{x_i,x_i}|)(\Delta))$
End
else $Mem := Mem + 1$
End

If $Mem = n$ then write ('Result is', $x_1,...,x_n$)
else if $k < S_{\max}$ then Begin $k := k+1$; go to 100 End
else write ('Approximative result is', $x_1,...,x_n$)

It is supposed that we use the following general equality:

$$M(|g|)(\Delta) = \max(|m(g)(\Delta)|, |M(g)(\Delta)|), \qquad \text{where } g \text{ is } f''_{x_i,x_i}.$$

Problem 2.2. Let $g, f_1, \ldots, f_k : D \to R$, where $D = [a_1, b_1] \times \ldots \times [a_n, b_n]$ be given m-M functions. Let A be the set of all points $(x_1, \ldots, x_n) \in D$ satisfying the inequalities

$$f_1(x_1, \ldots, x_n) \geq 0, \ldots, f_k(x_1, \ldots, x_n) \geq 0 \tag{2.9}$$

Restricting the function g to the set A we seek the set S of all points $(x_1, \ldots, x_n) \in A$ at wich g attains the minimum value (**the problem of constrained optimization** under the condition $(x_1, \ldots, x_n) \in D$).

In other words we seek all points $(x_1, \ldots, x_n) \in D$ satisfying the following conditions

1^0 $\quad f_1(x_1, \ldots, x_n) \geq 0, \ldots, f_k(x_1, \ldots, x_n) \geq 0$ (2.10)

2^0 $\quad (\forall (y_1, \ldots, y_n) \in D)[\, f_1(x_1, \ldots, x_n) \geq 0, \ldots, f_k(x_1, \ldots, x_n) \geq 0$
$\qquad \Rightarrow g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)]$

However, (2.10) treated as a conjuction of its parts 1^0 and 2^0 is *not* a $\leq$ positive formula, since 2^0 is logically equivalent to this formula

$$(\forall (y_1, \ldots, y_n) \in D)[\, f_1(y_1, \ldots, y_n) < 0 \vee, \ldots, \vee f_k(y_1, \ldots, y_n) < 0 \vee g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)]$$

Therefore in order to solve Problem 2.2 we cannot apply a procedure based on the equality of type (2.7). In connection with this obstacle we put the following assumption

If at some point $(x_1, \ldots, x_n) \in D$ the inequalities (*1)

$$f_1(x_1, \ldots, x_n) \geq 0, \ldots, f_k(x_1, \ldots, x_n) \geq 0$$

are satisfied then in each neighbourhood $N(x_1, \ldots, x_n)$ of this point there is a point $(x_{1o}, \ldots, x_{no}) \in D$ satisfying the inequalities

$$f_1(x_{1o}, \ldots, x_{no}) > 0, \ldots, f_k(x_{1o}, \ldots, x_{no}) > 0$$

In [1] using this assumption the following equivalence[8]

$$(\forall y_1) \ldots (\forall y_n)[\, (\forall i) f_i(y_1, \ldots, y_n) \geq 0 \Rightarrow g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)\,] \tag{*2}$$

is equivalent to the following $\leq$ formula

$$(\forall y_1) \ldots (\forall y_n)[(\forall i) f_i(y_1, \ldots, y_n) > 0 \Rightarrow g(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n)]$$

[8] Instead of $(\forall y_j \in [a_j, b_j])$, $(\forall i \in \{1, \ldots, k\})$ we write shortly $(\forall y_j)$, $(\forall i)$ respectively.

is proved. Consequently, under assumption (*1) we have the following definition of feasible Cartesian products of r-cells:

A Cartesian product $X_1 \times \ldots \times X_n$ of r-cells X_i is feasible (2.11)
if it satisfies the following conditions[9]

(i) $(\forall i \in \{1,\ldots,k\}) M(f_i)\,(X_1 \times \ldots \times X_n) \geq 0$

(ii) $(\forall Y_1 \in \mathcal{D}_r([a_1,b_1])) \ldots (\forall Y_n \in \mathcal{D}_r([a_n,b_n]))$
$(\forall i \in \{1,\ldots,k\})\, m^{10}(f_i)\,(Y_1 \times \ldots \times Y_n) > 0$
$\Rightarrow m(g)(X_1 \times \ldots \times X_n) \leq M(g)(Y_1 \times \ldots \times Y_n)$

It is very important that similarly as in Problem 2.1 we can use the idea (*5) from Problem 2.1 (where the symbol f is replaced by the symbol g). Further, under certain conditions we may add new requirements to the feasibility criterion. So, suppose that each of the function $f_1,\ldots,f_k,g$ satisfies a condition of type (2.8) and the function g attains the minimum value at some point, which is an internal point of the set A. Then to the conditions (2.11) 1^0, 2^0 we may add the following equations $\frac{\partial g}{\partial x_1} = 0, \cdots, \frac{\partial g}{\partial x_n} = 0$. Accordingly, the feasibility criterium should have also these requirements

$$m\left(\frac{\partial g}{\partial x_i}\right)(X_1 \times \cdots \times X_n) \leq 0 \leq M\left(\frac{\partial g}{\partial x_i}\right)(X_1 \times \cdots \times X_n) \quad (i = 1,\ldots,n)$$

Remark 2.2. If we replace (2.9) in Problem 2.2 by the following disjunction

$$f_1(x_1,\ldots,x_n) \geq 0 \vee \ldots \vee f_k(x_1,\ldots,x_n) \geq 0$$

we get a new problem which can be solved in a similar way as Problem 2.2 (such a problem belongs to the disjunctive-optimization problems).

Problem 2.3. We get this problem[11] from Problem 2.2 by replacing (2.9) by the following conditions

$$f_1(x_1,\ldots,x_n) = 0,\ldots,f_s(x_1,\ldots,x_n) = 0 \qquad (2.12)$$
$$f_{s+1}(x_1,\ldots,x_n) \geq 0,\ldots,f_k(x_1,\ldots,x_n) \geq 0 \quad (s \geq 1, k \geq s)$$

---

[9] In the formulation of (ii) the tautology $(\neg p \vee q) \Leftrightarrow (p \Rightarrow q)$ is employed.

[10] By mistake on this place in [1] stands M.

[11] This is Problem 5.3 from [1]. There are some mistakes in it. For instance in the formula (*2) the symbol = should be replaced by the symbol $\Rightarrow$. Next, in the last line of the condition (5.20) the symbol $\geq$ should be twice replaced by the symbol $>$. Also in the definition (5.22) the part $(\forall Y \in Sol_r(D))(M(f_{s+1})(Y) > 0,\ldots,M(f_k)(Y) > 0 \Rightarrow m(g)(X) \leq M(g)(Y))$ should be replaced by $(\forall Y \in Sol_r(D))(m(f_{s+1})(Y) > 0,\ldots,m(f_k)(Y) > 0 \Rightarrow m(g)(X) \leq M(g)(Y))$. Finally, the first sentence after (5.21), i.e. the sentence :"It is important that the second part ... " should be omitted.

Now the set A is defined as the set of all points $(x_1,\dots,x_n) \in D$ satisfying (2.12). This problem is more complicated than Problem 2.2. We shall describe two solving ideas.

The first idea. Denote by $Sol_r(D)$ the set of all $Y \in \mathcal{D}_r(D)$ which have at least one solution of the equations $f_1(x_1,\dots,x_n)=0,\dots,f_s(x_1,\dots,x_n)=0$, Suppose that

For each $r \geq r^0$ (r^0 is a constant) we can determine the set $Sol_r(D)$. (*1)

Further, on conditions (2.12) we put the following assumption (like (*1) in Problem 2.2.).

If $k > s$ and at some point $(x_1,\dots,x_n) \in D$ the formulas (*2)

$$f_1(x_1,\dots,x_n)=0,\dots,f_s(x_1,\dots,x_n)=0,\ f_{s+1}(x_1,\dots,x_n)\geq 0,\dots,f_k(x_1,\dots,x_n)\geq 0$$

are satisfied then in each neighbourhood $N(x_1,\dots,x_n)$ of this point there is a point $(x_{1o},\dots,x_{no}) \in D$ satisfying the formulas

$$f_1(x_{1o},\dots,x_{no})=0,\dots,f_s(x_{1o},\dots,x_{no})=0,\ f_{s+1}(x_{1o},\dots,x_{no})>0,\dots,f_k(x_{1o},\dots,x_{no})>0.$$

Using this assumption one can prove the following equivalence (like (*2) in Problem 2.2).

$$(\forall(y_1,\dots,y_n)\in D)[f_1(y_1,\dots,y_n)=0,\dots,f_s(y_1,\dots,y_n)=0,\ f_{s+1}(y_1,\dots,y_n)\geq 0,\dots,$$
$$f_k(y_1,\dots,y_n)\geq 0 \Rightarrow g(x_1,\dots,x_n)\leq g(y_1,\dots,y_n)]$$

is equivalent to

$$(\forall(y_1,\dots,y_n)\in D)[f_1(y_1,\dots,y_n)=0,\dots,f_s(y_1,\dots,y_n)=0,\ f_{s+1}(y_1,\dots,y_n)>0,\dots,$$
$$f_k(y_1,\dots,y_n)>0 \Rightarrow g(x_1,\dots,x_n)\leq g(y_1,\dots,y_n)]$$

Then under the conditions (*1), (*2) for each $r \geq r^0$ we have the following feasibility definition

An element $X \in Sol_r(D)$ is feasible if it satisfies the following condition (2.13)

$$(\forall Y \in Sol_r(D))(m(f_{s+1})(Y)>0,\dots,(m(f_k)(Y)>0 \Rightarrow m(g)(X)\leq M(g)(Y))$$

It is not difficult to prove that the set of all solutions of Problem 2.3 is equal to $\bigcap_{r\in N} F_r$. Also, we can use the idea like (*5) in Problem 2.1.

However, the main problem is how to determine $Sol_r(D)$. In other words how to find a condition $Cond(\Delta)$ such that the equivalence

$$\begin{array}{ll} \text{A cell } \Delta \text{ has at elast one} & \\ \text{solution of the system} & \Leftrightarrow \quad Cond(\Delta) \\ f_1(x_1,\dots,x_n)=0,\dots,f_s(x_1,\dots,x_n)=0 & \end{array}$$

is true. Roughly speaking, some "parts" of $Cond(\Delta)$ may be, for instance:

(j) Δ is "small enough".

(jj) For each of the functions $f_i (1 \le i \le s)$ there are two vertices V_1, V_2 of Δ such that $f_i(V_1) \cdot f_i(V_2) \le 0$.

The second idea. By this idea Problem 2.3 can be solved approximatively. Namely, let $\varepsilon > 0$ be a given "small" real number. Replacing (2.12) by the following inequalities

$$\begin{gathered} f_1(x_1,\dots,x_n) \ge -\varepsilon, \quad f_1(x_1,\dots,x_n) \le \varepsilon,\dots, f_s(x_1,\dots,x_n) \ge -\varepsilon, \quad f_s(x_1,\dots,x_n) \le \varepsilon \\ f_{s+1}(x_1,\dots,x_n) \ge 0,\dots, f_k(x_1,\dots,x_n) \ge 0 \end{gathered}$$

from Problem 2.3 we obtain a problem of type as Problem 2.2.

Problem 2.4. This problem belongs to the **Interval Mathematics.** Namely, consider problem of type (2.4) supposing that in the formula φ some constants $c_1, c_2,\dots,c_k$ appear which we do not know exactly. Instead, we are given certain constants L_i, R_i such that $L_i \le c_i \le R_i \quad (i = 1,2,\dots,k)$. So we have the problem

Solve $\phi(x_1,\dots,x_m)$ in $x_1 \in I(x_1),\dots,x_n \in I(x_n)$ where the constants $c_1,\dots,c_k$ satisfy the boundaries $L_i \le c_i \le R_i \quad (i = 1,2,\dots,k)$ (2.14)

As we shall see any such problem can be translated to a problem of type (2.4). To prove this, let us first consider problem (2.14) in case when $m = 0$. For instance, such a problem is stated in

Example 2.3. Examine the truth of the formula $(\forall x \in [1.4,\ 1.5])x^2 \ge 1.8...$ where 1.8... is a constant satsifying the boundaries $1.8 \le 1.8... \le 1.9$

Solution. Obviously this problem is logically equivalent to the following problem of type (2.4) with $m = 0$

Is the formula $(\forall c \in [1.8,\ 1.9])(\forall x \in [1.4,\ 1.5])x^2 \ge c$ true or false.

In general a problem:

Is the formula $\varphi(c_1,\dots,c_k)$ with boundaries $L_i \le c_i \le R_i$ true or false

is logically equivalent to the problem

Is the formula $(\forall c_1 \in [L_1, R_1])(\forall c_k \in [L_k, R_k])\ \varphi(c_1,\dots,c_k)$ true or false.

Now we shall consider problem (2.14) in case $m > 0$. For instance, such a problem is encountered in

Example 2.4. Find $x \in [1, 2]$ such that $x^2 = c$, where c is a constant with these information $1.69 \le c \le 1.96$ only.

Solution. Obviously the best information on x is expressed by the inequalities $1.3 \le x \le 1.4$. This conclusion can be divided in the following two implications

The first reads:

For all $c \in [1.69, 1.96]$ the implication $x^2 = c,\ x \in [1, 2] \Rightarrow 1.3 \le x \le 1.4$ (*)
is true.

The second reads:

If x, with $1.3 \le x \le 1.4$, is any number then for some $c \in [1.69, 1.96]$ (**)
the conditions $x^2 = c,\ x \in [1, 2]$ are true.

Notice that about (*) we have the following reformulations

$$(*) \quad \Leftrightarrow (\forall c \in [1.69, 1.96])\ (x^2 = c,\ x \in [1, 2] \Rightarrow 1.3 \le x \le 1.4)$$
$$\Leftrightarrow (\exists c \in [1.69, 1.96])\ (x^2 = c,\ x \in [1, 2]) \Rightarrow 1.3 \le x \le 1.4$$

(By applying the following logically valid formula
$$(\forall c)(\alpha(c) \Rightarrow \beta) \Leftrightarrow ((\exists c)\alpha(c) \Rightarrow \beta)$$
provided that c is not a free variable in β).

On other hand, about (**) we have

$$(**) \quad \Leftrightarrow (1.3 \le x \le 1.4 \Rightarrow (\exists c \in [1.69, 1.96])\ (x^2 = c,\ x \in [1, 2])$$

Combining the obtained results we have the following equivalence

$$1.3 \le x \le 1.4 \Leftrightarrow (\exists c \in [1.69, 1.96])\ x^2 = c,\ x \in [1, 2]$$

Consequently we have the following conclusion. The problem stated in Example 2.4 is logically equivalent to the following problem:

Find $x \in [1, 2]$ such that the formula $(\exists c \in [1.69, 1.96])\ x^2 = c$ is true.

The reasoning employed in this example can be transfered to any (2.14) problem with given boundaries $L_i \le c_i \le R_i$. Namely:

Any (2.14) ($m > 0$)problem with given boundaries $L_i \le c_i \le R_i \, (i = 1,...,k)$ is logically equivalent to the following problem of type (2.4):

Find all values of $x_i \in I(x_i) \quad (i = 1,...,k)$ such that the formula

$$(\exists c_1 \in [L_1, R_1]) \ldots (\exists c_k \in [L_k, R_k]) \; \varphi(c_1,...,c_k)$$

is satisfied.

Example 2.5. Find $z \in [-7,25]$ such that the condition $(\forall x \in [0,4])$ $(\exists y \in [3,5])$ $y^2 - x^2 = z$ is satisfied.

Solution. The ordinary binary trees are used. The calculations are carried out only up to step 7. The number of feasible cells is 2 in each step, and z satisfies: $8.75 \le z \le 9.25$.

3. FINDING FUNCTIONS AS SOLUTIONS OF A GIVEN m-M CONDITIONS

Let $\varphi(x, y)$ be a positive $\le$ formula, quantifier-free, and whose free variables are x, y. Replacing y by a term $f(x)$, where f is a function symbol from the formula $\varphi(x, y)$ we obtain

$$\varphi(x, f(x)) \tag{3.1}$$

which we shall call "an m-M (functional) condition". Let $A, B \subset R$ be given segments and $f : A \to B$ a function satisfying the condition $(\forall x \in A) \, \varphi(x, f(x))$. Then we say that f is a solution of condition (3.1). In the sequel we are going to describe a procedure by which one can step-by-step approximatively determine all such functions (if any exists). We shall use the following denotations

* X will be a sequence of some subsegments (i.e. cells) of the segment A. By $l(X)$ is denoted the number of its elements.
* If $P \in X$ then by $F(P)$, $F_1(P)$ will be denoted some sequences of subsegments of the segment B; $l(F(P))$, $l(F_1(P))$ are the numbers of their elements.

We also use the following convention:

Two segments of the forms $[p,q]$, $[r,s]$ are called **neighbouring** if $q = r$ or $s = p$.

In fact in the procedure we search certain solution $x \in A$, $y \in B$ of $\varphi(x,y)$, having in mind that y should be a function of x. The procedure reads:

(i) If $m(\varphi)(A \times B)$ is false the procedure stops and the result is: (3.2)

(3.1) has none f-solution.

In the opposite case we take:

$$l(X) = 1,\ X_1 = A,\ l(F(A)) = 1,\ B \text{ is the unique element of } F(A);$$

and go to (ii).

(ii) In turn we take $P = X_i \quad (1 \le i \le l(X))$ and for each of them we do the following:

From sequence $F(P)$ we form a new sequence $F_1(P)$ consisting of all elements $Q \in F(P)$ for which the condition $m(\varphi)(P \times Q)$ holds. If the sequence $F_1(P)$ is empty then the procedure stops and the result is: (3.1) has none f-solution.

In the opposite case we first make unions of all neighbouring elements of the sequence $F_1(P)$ and in such a way we obtain a new sequence, which we call $F_1(P)$ again[12].

After $P = X_{l(X)}$ is being processed we go to (iii).

(iii) In this step we have already determined the desired function f approximatively

Namely, for any $x \in A$ let $P \subset X$ be a segment containing this x. Then $f(x)$ may be any number which is any element of some element of $F_1(P)$.

If we want to continue the procedure then we do the following:

First, for each $P = X_i \quad (1 \le i \le l(X))$ we do the following:

We decompose P into smaller subsegments, say $P_1, \ldots, P_r$ and temporarily extend the function F_1 by the conditions

$$F_1(P_1) = \ldots = F_1(P_r) = F_1(P)$$

Let X_1 be the sequence of all such subsegments for all elements $P \in X$. Next, in turn to each element $P \in X_1$ we consider the related sequence $F_1(P)$ and decompose all its elements into some smaller subsegments. In such a way from $F_1(P)$ we obtain a new sequence named $F(P)$. We put $X = X_1$ and go to (ii).

---

[12] For instance, if $F_1(P)$ is the sequence [1,2], [7,8], [2,3], [6,7], [3,4], [9,10] then the new $F_1(P)$ is the following sequence [1,4], [6,8], [9,10].

In a similar way one can approximatively solve any functional condition like (3.1) under this restriction:

All unknown functions have the same number of arguments.

For instance, the functional conditions $\varphi(x,f(x),g(x))$, $\psi(x,y,h(x,y),k(x,y),m(x,y))$ where f,g,h,k,m are unknown functions belong to this class. However, the functional condition

$$\varphi(x,f(x),y,g(x,y)) \tag{3.3}$$

obviously is not a member of the class. Solving procedure for such conditions in some details differs from Procedure (3.2). For example to solve (3.3) we proceed as follows:

We replace (3.3) by the following functional condition $\varphi(x,f_1(x,y),y,g(x,y))$

with two unknown functions f_1,g. Then using a Procedure like (3.2) we seek those of its solutions which are solutions of (3.3) too. Namely, suppose that in some step for x and y we have all together the following subsegments

$$X_1,\dots,X_p,\ Y_1,\dots,Y_q$$

respectively. Let $P=X_i$ be any of these $X_1,\dots,X_p$. To define the sequence $F(P)$ we consider all sequences

(*) $\quad F_1(P,Y_1),\dots,F_1(P,Y_q)$

and then: any subsegment $[a,b]$ is an element $F(P)$ if and only if this subsegment belongs to each member of the sequences (*). If $F(P)$ is empty sequence the procedure stops with the conclusion that (3.3) has no solutions.

Let now $\varphi(x,y,z)$ be a positive $\leq$ formula, quantifier-free, whose all free variables are x,y,z. Replacing y,z by the following terms f_x,f_{x+h} respectively from the formula $\varphi(x,y,z)$ we obtain

$$\varphi(x,f_x,f_{x+h}) \qquad (h \text{ is a given positive constant}) \tag{3.4}$$

which we are going to call "an m-M difference condition". Concerning (3.4) the problem is:

Let a,a',b (with $a<b$) be some given real numbers. Giving to x and f_x initial values a,a' respectively determine a finite sequences (if any exists) (3.5)

$$f_a,f_{a+h},\dots,f_{a+nh} \qquad (a+nh \text{ is } b)$$

such that (3.4) is satisfied for every $x\in\{a,a+h,\dots,a+(n-1)h\}$.

In the sequel we shall describe a procedure by which one can approximatively determine all such sequences[13], under the restriction that $f_a, f_{a+h}, \dots, f_{a+nh}$ belong to a given segment $B \subset R$.

As a matter of fact problem (3.5) is logically equivalent to the following:

Solve for $f_{a+h}, \dots, f_{a+nh} \in B$ the system (3.6)

$$\begin{aligned}&\varphi(a, a', f_{a+h})\\&\varphi(a+h, f_{a+h}, f_{a+2h})\\&\vdots\\&\varphi(a+(n-1)h, f_{a+(n-1)h}, f_{a+nh})\end{aligned}$$

The procedure reads:

Using a procedure like Procedure 1.1 we approximatively solve for[14] f_{a+h} the first formula, i.e. the formula $\varphi(a, a', f_{a+h})$. In such a way we obtain as the result some set F_{a+h}-union of some subsegments of the set B. Next, we go to the second formula $\varphi(a+h, f_{a+h}, f_{a+2h})$ which we shall solve for f_{a+2h} under the assumption $f_{a+2h} \in F_{a+h}$. In other words[15], we need to solve for $f_{a+2h} \in B$ this formula (3.7)

$$(\exists y \in F_{a+h})\varphi(a+h, y, f_{a+2h})$$

Again we apply a corresponding procedure[16] and for f_{a+2h} we determine some set f_{a+2h}-union of some subsegments[17] of the set B. Similarly we proceed with the remaining formulas $\varphi(a+2h, f_{a+2h}, f_{a+3h}), \dots, \varphi(a+(n-1)h, f_{a+(n-1)h}, f_{a+nh})$. So, solving the formula

$$(\exists y \in F_{a+2h})\varphi(a+2h, y, f_{a+3h})$$

we obtain the set F_{a+2h}; solving the formula

$$(\exists y \in F_{a+3h})\varphi(a+3h, y, f_{a+4h})$$

we obtain the set F_{a+3h}, and so on. Finally, the desired sequence is approximatively determined by these conclusions:

$$f_a = a', f_{a+h} \in F_{a+h}, \dots, f_{a+nh} \in F_{a+nh}$$

---

[13] if any exists

[14] This means that we use the solving procedure up to some step r, and this r is of our choice.

[15] See Problem 2.4, Example 2.4.

[16] Up to some step r, r is a number of our choice.

[17] In fact, they are corresponding feasible cells.

Of course, it can happen that for some i the set F_i is empty, when the procedure should be stopped with the conclusion that the desired sequence does not exist.

We point that besides (3.4) one can in a similar way solve various other difference conditions like $\varphi(x, f_x, f_{x+h}, f_{x+2h})$, and so on.

In this part we state a procedure by which one can approximatively solve a given differential equation. Let

$$E(x, f(x), f'(x)) = 0 \tag{3.8}$$

be a given differential equation having a solution $f : A \to B$. Denote by C a set with the property

$$f'(x) \in C \qquad \text{whenever } x \in A$$

Suppose that there exists $f''(x)$ for $x \in A$, and that the function $E(x,y,z)$ (with $x \in A$, $y \in B$, $z \in C$) is differentiable. Additionally suppose

There are positive constants K_1, K_2 such that (3.9)

$$\left|\frac{\partial E}{\partial z}\right| \le K_1, \quad |f''(x)| \le K_2$$

for every $x \in A$, $y \in B$, $z \in C$.

Then one can immediately prove the following assertion

If x and $x+h$ (with $h > 0$) are any elements of A then the inequality

$$\left| E\left(x, f(x), \frac{f(x+h) - f(x)}{h}\right) \right| \le K_1 K_2 h \tag{3.10}$$

holds.

From inequality (3.10) one can easily get an idea for solving equation (3.8). Namely, first suppose $E(x,y,z)$ is an m-M function. Then to (3.10) one can assign the corresponding m-M difference condition, say expressed in this manner[18]

$$\left| E\left(x, f(x), \frac{1}{h}(f_{x+h} - f_x)\right) \right| \le K_1 K_2 h \tag{3.11}$$

[18] Now, f is used as a sequence-symbol.

In such a way we obtain an example of the difference condition of type (3.4). Consequently we can apply a procedure of type[19] (3.7). Of course, in order to do this we have to know the constants K_1, K_2 in advance. About K_1 it suffices to suppose that $\frac{\partial E}{\partial z}$ is an m-M function. To find K_2, briefly said, one can besides equation (3.8) employ the equation

$$\frac{\partial E}{\partial z}+\frac{\partial E}{\partial y}f'_y+\frac{\partial E}{\partial z}f''_z=0$$

At the end we emphasize that by solving difference condition (3.11) we in fact approximatively determine all solutions of differential equation (3.8) with initial condition $f(a)=a'$. But, if solving procedure halts then we conclude that equation (3.8) has none such solution.

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[19] The numbers a,b are determined by the set A, while a', h (with $a' \in B$, $h > 0$) are of our choice.